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J. M. De Villiers

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ASYMPTOTIC SOLUTIONS OF THE ORR–SOMMERFELD EQUATION

By J. M. DE VILLIERS†
Department of Applied Mathematics and Theoretical Physics, University of Cambridge

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† Present address: Department of Applied Mathematics, University of Stellenbosch, Stellenbosch 7600, South Africa.

A rigorous justification is given of work done by Eagles (1969), in which he applied the method of matched asymptotic expansions to the Orr–Sommerfeld equation to obtain *formal* uniform asymptotic approximations to a certain pair of solutions. (Somewhat more polished formal expansions of the same general kind were subsequently obtained by Reid (1972).) First, a study is made of the asymptotic properties of solutions of a certain differential equation which admits the Orr–Sommerfeld equation as a special case. Previous work on this differential equation by Lin & Rabenstein (1960, 1969) is extended to develop a theory suited to our main purpose: to prove the validity of Eagles's approximations. It is then shown how this theory can be used to prove the existence of *actual* solutions of the Orr–Sommerfeld equation approximated by these formal expansions. In addition, it is verified that these solutions have the properties assumed by Eagles (1969).

1. Introduction

In this paper we consider the Orr–Sommerfeld equation

$$\frac{d^4\Phi}{dy^4} - 2k^2\frac{d^2\Phi}{dy^2} + k^4\Phi - ikR\left\{(w(y)-c)\left(\frac{d^2\Phi}{dy^2} - k^2\Phi\right) - w''(y)\,\Phi\right\} = 0 \tag{1.1}$$

in a complex neighbourhood of [0, 1], and study the asymptotic forms of solutions for large kR. Here $\Phi(y)\,e^{ik(x-ct)}$ is the stream function of a small disturbance with a real non-negative wave number k, $w(y)$ is the basic velocity profile, and R is the Reynolds number.

Eagles (1969) applied the widely used method of 'matched asymptotic expansions' (Fraenkel 1969) to construct, in a systematic manner, *formal* uniform asymptotic approximations to a certain pair of solutions of (1.1), which he called Φ_5 and Φ_3, and which he assumed to have certain properties. The principal aim of the present paper is to prove the validity of Eagles's formal approximations. To achieve this, we use the asymptotic solutions of Lin & Rabenstein (1960, 1969), the construction of which is rigorous but uses more elaborate and specialized methods, and a more complicated comparison equation, than Eagles needed to construct his formal approximations.

First, in § 2 below, we show how the theory of Lin & Rabenstein can be used to obtain four linearly independent asymptotic solutions of a certain differential equation which is less general than that studied by Lin & Rabenstein (1960, 1969), but admits equation (1.1) as a special case. We have the following specific reasons for giving (in §§ 2 (*a*), (*b*)) a detailed account of those elements of the theory of Lin & Rabenstein (1960) which are relevant in our case.

(*a*) To save the reader need for repeated reference to Lin & Rabenstein (1960).

(*b*) In contrast to Lin & Rabenstein (1960), we use only *even* powers of λ^{-1} (= const. × $(kR)^{-\frac{1}{3}}$ in the Orr–Sommerfeld case) in the transformations (2.5) and (2.15). The resulting theory, although not as general as that of Lin & Rabenstein (1960), is simpler, and suffices for our final (Orr–Sommerfeld) application.

(*c*) The extended use of vector notation adopted here, the correction of a substantial number of minor errors in Lin & Rabenstein's (1960) paper, and the sharper statement of certain results, perhaps clarifies the theory.

(*d*) Our final application requires not merely the final form of the Lin–Rabenstein solutions, but certain steps in the derivation of those solutions.

The results of § 2 enable us to develop a theory which eventually leads (in § 6) to our main result: that there do indeed exist solutions Φ_5 and Φ_3 of (1.1) with the properties assumed by Eagles, and which are approximated by his formal expansions.

It seems likely that the solutions in §§4 and 5 would also be of use in any attempt to justify rigorously the approximations of Reid (1972), but this task would also require a substantial amount of further work (cf. §6 and appendix B).

2. Implications of the theory of Lin & Rabenstein

(a) *Preliminaries and an outline of* §2

We consider the equation

$$\mathscr{L}\Phi \equiv \frac{d^4\Phi}{ds^4} + \lambda^2\left\{P(s,\lambda)\frac{d^2\Phi}{ds^2} + R(s,\lambda)\,\Phi\right\} = 0 \quad \text{in} \quad D_1, \tag{2.1}$$

and study its asymptotic solutions for large $|\lambda|$. Here D_1 is a bounded domain in the complex plane $\boldsymbol{C}$ containing a complex neighbourhood of the origin, and the coefficients $P(s,\lambda)$ and $R(s,\lambda)$ are holomorphic functions of s in D_1, and holomorphic functions of λ outside a sufficiently large disk (about the origin). Furthermore,

$$P(s,\lambda) \sim \sum_{n=0}^{\infty} \lambda^{-2n} P_{2n}(s), \quad R(s,\lambda) \sim \sum_{n=0}^{\infty} \lambda^{-2n} R_{2n}(s) \quad \text{for} \quad |\lambda| \to \infty,$$

with

$$P_0(0) = 0, \quad P_0'(0) = 1,$$

and

$$P_0(s) \neq 0, \quad \int_0^s P_0^{\frac{1}{2}}(s')\,ds' \neq 0 \quad \text{for} \quad s \in D_1\backslash\{0\}.$$

The assumption that $P_0(s)$ has a simple zero at the origin makes the point $s = 0$ a so-called *turning point* (of the first order) of $\mathscr{L}\Phi = 0$. The significance of the turning point is that the *reduced equation*

$$P_0(s)\,d^2\Phi/ds^2 + R_0(s)\,\Phi = 0,$$

which is obtained by formally letting $\lambda \to \infty$ in $\mathscr{L}\Phi = 0$, has a regular singular point at $s = 0$ (except in the special case $R_0(0) = 0$).

Definitions. (*a*) Given a bounded domain Ω in $\boldsymbol{C}$, we denote by $\mathscr{H}(\Omega)$ the class of all scalar-, vector- and matrix-valued functions which are holomorphic in Ω.

(*b*) We shall say that a scalar-, vector- or matrix-valued function f belongs to the class Λ if and only if

$$f(z,\lambda) \sim \sum_{n=0}^{\infty} \lambda^{-2n} f_{2n}(z) \quad \text{for} \quad |\lambda| \to \infty,$$

or belongs to the class $\Lambda(2m)$ if and only if

$$f(z,\lambda) = \sum_{n=0}^{m} \lambda^{-2n} f_{2n}(z),$$

where each $f_{2n} \in \mathscr{H}(D_2)$, and D_2 is the image of D_1 under the mapping (2.2*a*) below.

Now introduce

$$z(s) = \left\{\frac{3}{2}\int_0^s P_0^{\frac{1}{2}}(s')\,ds'\right\}^{\frac{2}{3}}, \tag{2.2a}$$

$$\psi(z) = (dz/ds)^{\frac{3}{2}}\,\Phi(s). \tag{2.2b}$$

It is clear that the transformation (2.2*a*) maps the turning point $s = 0$ on to $z = 0$, and maps D_1 on to a bounded domain D_2 (say) in the z-plane. Furthermore, (2.2*a*, *b*) transforms $\mathscr{L}\Phi = 0$ to

$$\mathscr{M}\psi \equiv \psi^{\text{iv}} + \lambda^2\{p(z,\lambda)\,\psi'' + q(z,\lambda)\,\psi' + r(z,\lambda)\,\psi\} = 0 \quad \text{in} \quad D_2, \tag{2.3a}$$

where we now write $(.)'$ for $(\mathrm{d}/\mathrm{d}z)\,(.)$, and where $p, q, r \in \Lambda$, with

$$p_0(z) = z, \quad q_0(0) = 0, \quad r_0(0) = R_0(0). \tag{2.3b}$$

Henceforth we adopt the *vector notation*

$$\boldsymbol{v} = \begin{bmatrix} v \\ v' \\ \lambda^{-2}v'' \\ \lambda^{-2}v''' \end{bmatrix}$$

for column vectors. In vector form, the equation $\mathscr{M}\psi = 0$ becomes

$$\boldsymbol{\psi}' = \boldsymbol{M}\boldsymbol{\psi},$$

where
$$\boldsymbol{M} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 1 \\ -r & -q & -\lambda^2 p & 0 \end{bmatrix}. \tag{2.4}$$

In §2(*b*), we introduce the scalar

$$\phi = \boldsymbol{A}(z,\lambda).\boldsymbol{\psi} = A(z)\,\psi + B(z,\lambda)\,\psi' + C(z)\,\lambda^{-2}\psi'' + D(z)\lambda^{-2}\psi''', \tag{2.5}$$

and show that we can choose the row vector $\boldsymbol{A}(z,\lambda) = (A, B, C, D) \in \Lambda(2)$ in such a way that the equation $\mathscr{M}\psi = 0$ transforms to the *adjoint normal form*†

$$\mathscr{N}^*\phi = \phi^{\mathrm{iv}} + \lambda^2\{z\phi'' + \alpha^*\phi' + \beta^*\phi\} - \boldsymbol{a}_*(z,\lambda).\boldsymbol{\phi} = 0 \quad \text{in} \quad D_2, \tag{2.6}$$

where α_* and β_* are certain constants, and where the vector $\boldsymbol{a}_*(z,\lambda) = (a_*, b_*, c_*, d_*) \in \Lambda$.

Remarks. (*a*) The basic reason for transforming $\mathscr{M}\psi = 0$ to $\mathscr{N}^*\phi = 0$ by means of (2.5), is that this leads us to a study of the equation $\mathscr{N}\chi = 0$ below, and approximate solutions $\chi_{(2m)}$ of this equation can be constructed from the *known* asymptotic solutions (Rabenstein 1958) of the basic reference equation $\mathscr{Q}u = 0$ given by (2.12) below.

(*b*) It might seem more natural to adopt a transformation $\psi = \tilde{\boldsymbol{A}}.\boldsymbol{\phi}$, but the form (2.5) adopted here has certain advantages. In the case $\psi = \tilde{\boldsymbol{A}}.\boldsymbol{\phi}$, for example, the analogue of (2.18*a*, *b*) below would contain the vector $\tilde{\boldsymbol{A}}$ and the matrix $\boldsymbol{N}^*$, *both* of which still has to be determined at that stage. The transformation $\phi = \boldsymbol{A}.\boldsymbol{\psi}$, however, immediately leads to the relations (2.18*a*, *b*) which contain $\boldsymbol{A}$ and the *known* matrix $\boldsymbol{M}$. The vector $\boldsymbol{A}$ is then *first* subjected to the conditions (2.20) and (2.23), and *only then* it becomes necessary to introduce $\mathscr{N}^*\phi = 0$ in order to specify $\boldsymbol{A}$ further.

Note that the equation $\mathscr{N}^*\phi = 0$ can equivalently be written in the vector form

$$\left.\begin{aligned} &\boldsymbol{\phi}' = \boldsymbol{N}^*\boldsymbol{\phi}, \\ &\text{where} \quad \boldsymbol{N}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta_* + \lambda^{-2}a_* & -\alpha_* + \lambda^{-2}b_* & -\lambda^2 z + \lambda^{-2}c_* & \lambda^{-2}d_* \end{bmatrix}. \end{aligned}\right\} \tag{2.7}$$

† Although the *normal* form of an nth order equation usually refers to the situation where the $(n-1)$st derivative is absent, or has been eliminated by a preliminary transformation, we shall follow Lin & Rabenstein (1960) in calling equation (2.6) a normal form (in spite of it containing the term $-d_*\lambda^{-2}\phi'''$).

However, in order to make the theory of Lin & Rabenstein (1969) applicable, it is necessary to consider the equation

$$\mathcal{N}\chi \equiv \chi^{\mathrm{iv}} + \lambda^2\{z\chi'' + \alpha_0\chi' + \beta_0\chi\} - \boldsymbol{a}(z,\lambda).\boldsymbol{\chi} = 0 \quad \text{in} \quad D_2,$$

or equivalently, in vector form, $\boldsymbol{\chi}' = \boldsymbol{N\chi}$,

where

$$\boldsymbol{N} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta_0 + \lambda^{-2}a & -\alpha_0 + \lambda^{-2}b & -\lambda^2 z + \lambda^{-2}c & \lambda^{-2}d \end{bmatrix}. \tag{2.8}$$

We shall call $\mathcal{N}\chi = 0$ *the normal form*. Here $\mathcal{N}$ and $\mathcal{N}^*$ are (real-type) *adjoints*† of each other, so that

$$\left.\begin{aligned} &\alpha_0 = 2 - \alpha_*, \quad \beta_0 = \beta_*, \\ &a(z,\lambda) = a_* - b_*' + \lambda^{-2}(c_*'' - d_*'''), \quad b(z,\lambda) = -b_* + \lambda^{-2}(2c_*' - 3d_*''), \\ &c(z,\lambda) = c_* - 3d_*', \quad d(z,\lambda) = -d_*. \end{aligned}\right\} \tag{2.9}$$

Hence $\boldsymbol{a}(z,\lambda) = (a,b,c,d) \in \Lambda$. Note that the normal form $\mathcal{N}\chi = 0$ can also be written in the form

$$L_0\chi - \boldsymbol{a}(z,\lambda).\boldsymbol{\chi} = 0 \quad \text{in} \quad D_2, \tag{2.10}$$

where the *dominant differential operator* L_0 is defined by

$$L_0 u \equiv u^{\mathrm{iv}} + \lambda^2\{zu'' + \alpha_0 u' + \beta_0 u\}, \tag{2.11}$$

and will be prominent in the differential equations below.

In §2(*c*) we define the *basic reference equation*

$$\mathcal{Q}u \equiv u^{\mathrm{iv}} + \lambda^2\{zu'' + \alpha(\lambda)\,u' + \beta(\lambda)\,u\} = 0 \quad \text{in} \quad D_2, \tag{2.12}$$

where α and β are in $\Lambda(2m)$ for some non-negative integer m, and are independent of z. In vector form, $\mathcal{Q}u = 0$ becomes $\boldsymbol{u}' = \boldsymbol{Qu}$,

where

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & -\alpha & -\lambda^2 z & 0 \end{bmatrix}. \tag{2.13}$$

Note that $\mathcal{Q}u = 0$ can be written in the form

$$\left.\begin{aligned} &L_0 u + \gamma(\lambda).\boldsymbol{u} = 0 \quad \text{in} \quad D_2, \\ &\text{where} \quad \gamma(\lambda) = \begin{cases} \lambda^2(\alpha - \alpha_0, \beta - \beta_0, 0, 0) \in \Lambda(2m-2) & \text{if} \quad m \geqslant 1, \\ 0 \quad \text{if} \quad m = 0. \end{cases} \end{aligned}\right\} \tag{2.14}$$

We then show that it is possible to find coefficients $\alpha, \beta \in \Lambda(2m)$ for $\mathcal{Q}u = 0$, and a row vector $\boldsymbol{\mu}(z,\lambda) = (\mu,\nu,\sigma,\tau) \in \Lambda(2m+2)$ such that a solution u of $\mathcal{Q}u = 0$ generates, by means of the transformation

$$\chi_{(2m)} = \boldsymbol{\mu}(z,\lambda).\boldsymbol{u} = \mu u + \nu u' + \sigma\lambda^{-2}u'' + \tau\lambda^{-2}u''', \tag{2.15}$$

a formally approximate solution $\chi_{(2m)}$ of $\mathcal{N}\chi = 0$. More precisely, $\chi_{(2m)}$ satisfies a *related equation* of the form

$$L_0\chi_{(2m)} - \boldsymbol{a}(z,\lambda).\boldsymbol{\chi}_{(2m)} = \lambda^{-2m}\hat{\boldsymbol{a}}(z,\lambda).\boldsymbol{\chi}_{(2m)}, \tag{2.16}$$

where $\hat{\boldsymbol{a}}(z,\lambda) = (\hat{a},\hat{b},\hat{c},\hat{d}) \in \Lambda$, and where m is an arbitrary non-negative integer.

† We do not take complex conjugates in defining the adjoint of a linear differential operator (in contrast to Coddington & Levinson (1955), p. 84), because L^2-theory will not be needed.

The validity of the formal approximation $\chi_{(2m)}$ is proved by following Lin & Rabenstein (1969); in fact, as in the theorems of that paper we obtain a fundamental set of solutions, with given asymptotic properties for $|\lambda| \to \infty$, of the normal form $\mathcal{N}\chi = 0$. Finally, in § 2(*e*), we show how these solutions of $\mathcal{N}\chi = 0$, together with a fundamental set of solutions of the basic reference equation $\mathcal{Q}u = 0$ (studied by Rabenstein (1958) and given in appendix A), can be used to derive the asymptotic properties of a fundamental set of solutions of the adjoint normal form $\mathcal{N}^*\phi = 0$.

(*b*) *Transformation to the adjoint normal form* $\mathcal{N}^*\phi = 0$

We consider
$$\mathcal{M}\psi = 0 \quad \text{in} \quad D_2$$
as in (2.3*a*, *b*), adopt the vector notation
$$\boldsymbol{v} = \begin{bmatrix} v \\ v' \\ \lambda^{-2}v'' \\ \lambda^{-2}v''' \end{bmatrix} \tag{2.17}$$
for column vectors, and introduce the scalar
$$\phi = \boldsymbol{A}(z,\lambda).\boldsymbol{\psi} = A(z)\,\psi + B(z,\lambda)\,\psi' + C(z)\,\lambda^{-2}\psi'' + D(z)\,\lambda^{-2}\psi''',$$
with
$$B(z,\lambda) = B_0(z) + \lambda^{-2}B_2(z),$$
and where $\boldsymbol{A}(z,\lambda) = (A, B, C, D)$ is a row vector in $\Lambda(2)$ to be chosen. Using the vector form $\boldsymbol{\psi}' = \boldsymbol{M}\boldsymbol{\psi}$ of $\mathcal{M}\psi = 0$, we find that
$$(\mathrm{d}/\mathrm{d}z)^j\phi = \boldsymbol{A}^j.\boldsymbol{\psi} \quad (j = 0, 1, 2, 3, 4), \tag{2.18a}$$
where the superscript j of $\boldsymbol{A}^j$ is merely a *label* (it does *not* denote the jth power), and in fact
$$\boldsymbol{A}^0(z,\lambda) = \boldsymbol{A}(z,\lambda), \quad \boldsymbol{A}^{j+1}(z,\lambda) = (\boldsymbol{A}^j)' + \boldsymbol{A}^j\boldsymbol{M} \quad (j = 0, 1, 2, 3). \tag{2.18b}$$
Thus $\phi = \boldsymbol{A}.\boldsymbol{\psi}$ and $\boldsymbol{\psi}' = \boldsymbol{M}\boldsymbol{\psi}$ imply that
$$\boldsymbol{\phi} = \boldsymbol{G}\boldsymbol{\psi}, \tag{2.19}$$
where
$$\boldsymbol{G} = \begin{bmatrix} A(z) & B(z,\lambda) & C(z) & D(z) \\ A^1(z,\lambda) & B^1(z,\lambda) & C^1(z,\lambda) & D^1(z) \\ \lambda^{-2}A^2(z,\lambda) & \lambda^{-2}B^2(z,\lambda) & \lambda^{-2}C^2(z,\lambda) & \lambda^{-2}D^2(z,\lambda) \\ \lambda^{-2}A^3(z,\lambda) & \lambda^{-2}B^3(z,\lambda) & \lambda^{-2}C^3(z,\lambda) & \lambda^{-2}D^3(z,\lambda) \end{bmatrix}.$$
However, we demand that $\boldsymbol{G} \in \Lambda$. Therefore, since
$$\boldsymbol{A}^1 = \boldsymbol{A}' + \boldsymbol{A}\boldsymbol{M} = \boldsymbol{A}' + (-rD, A - qD, \lambda^2 B - \lambda^2 pD, C)$$
by (2.4), and since $p, q, r \in \Lambda$, we must have
$$B_0 = p_0 D \equiv zD \tag{2.20}$$
by (2.3*b*), and it follows that $\quad \boldsymbol{A}_0 = (A, B_0, C, z^{-1}B_0)$.

Repeated use of the formula (2.18*b*) yields
$$\boldsymbol{G}_0 = \begin{bmatrix} A & B_0 & C & z^{-1}B_0 \\ A_0^1 & B_0^1 & C_0^1 & D^1 \\ 0 & 0 & B_0^1 - zD^1 & 0 \\ 0 & 0 & \hat{C}_0^3 & B_0^1 - zD^1 \end{bmatrix}, \tag{2.21a}$$

where $$\mathbf{A}_0^j = (\mathbf{A}^j)_0 = \begin{cases} (A_0^j, B_0^j, C_0^j, D^j) & (j=1), \\ (A_0^j, B_0^j, C_0^j, D_0^j) & (j=2,3), \end{cases}$$

is the leading term in the expansion of $\mathbf{A}^j$ in powers of λ^{-2}, and

$$\left.\begin{aligned} A_0^1 &= A' - r_0 z^{-1} B_0, \\ B_0^1 &= B_0' + A - q_0 z^{-1} B_0, \\ C_0^1 &= C' + B_2 - p_2 z^{-1} B_0, \\ D^1 &= (z^{-1}B_0)' + C, \\ \hat{C}_0^3 &= 2(B_0^1)' - 2z(D^1)' - (q_0+1)\, D^1 + A_0^1 - zC_0^1. \end{aligned}\right\} \tag{2.21b}$$

The relations (2.21 b) show that we may regard the functions $A(z)$, $B_0(z)$, $C_0^1(z)$ and $D^1(z)$ as the undetermined elements of $\mathbf{G}_0$. It is clear that the matrix $\mathbf{G}$ will be known once $\mathbf{G}_0$ has been determined.

Consider the matrix $\mathbf{G}_0(z)$ at $z=0$. Since $B_0(z) = zD(z)$ by (2.20), we have $B_0(0) = 0$ and

$$(\det \mathbf{G}_0)(0) = A(0)\{B_0^1(0)\}^3; \tag{2.22}$$

therefore, *for the matrix* $\mathbf{G}$ *in the transformation* $\boldsymbol{\phi} = \mathbf{G}\boldsymbol{\psi}$ *to be invertible, we demand that*

$$A(0) \neq 0, \quad B_0^1(0) \neq 0. \tag{2.23}$$

So far, the vector $\mathbf{A}$ is subject only to the conditions $B_0 = zD$ and (2.23); we now specify it further to obtain the desired adjoint normal form $\mathcal{N}^*\phi = 0$. By the definition (2.6) of $\mathcal{N}^*$ we can write

$$\mathcal{N}^*\phi \equiv L_0^*\phi - \mathbf{a}_*(z,\lambda)\,.\,\boldsymbol{\phi},$$

where $$L_0^*\phi \equiv \phi^{\text{iv}} + \lambda^2\{z\phi'' + \alpha_*\phi' + \beta_*\phi\}.$$

Then, for *any* constants α_* and β_*, the transformation formula (2.18 a) implies that

$$L_0^*\phi = \mathbf{A}^4.\boldsymbol{\psi} + \lambda^2\{z\mathbf{A}^2.\boldsymbol{\psi} + \alpha_*\mathbf{A}^1.\boldsymbol{\psi} + \beta_*\mathbf{A}.\boldsymbol{\psi}\} \equiv \tilde{\mathbf{A}}.\boldsymbol{\psi},$$

where $$\tilde{\mathbf{A}} = \mathbf{A}^4 + \lambda^2\{z\mathbf{A}^2 + \alpha_*\mathbf{A}^1 + \beta_*\mathbf{A}\}.$$

We shall choose $A(z)$, $B_0(z)$, $C_0^1(z)$ and $D^1(z)$ in such a way that $\tilde{\mathbf{A}}$ belongs to Λ; in other words, we shall equate to zero the coefficient of λ^2 in the expansion of $\tilde{\mathbf{A}}$. It can be verified, using the relations (2.18 b), that $\tilde{\mathbf{A}} = \lambda^2\hat{\mathbf{A}}$, where $\hat{\mathbf{A}} \in \Lambda$. Hence, if the requirement $\hat{\mathbf{A}}_0 = 0$, as well as the condition (2.23) can be realized, it will follow that $\tilde{\mathbf{A}} \in \Lambda$, and consequently that

$$\begin{aligned} L_0^*\phi = \tilde{\mathbf{A}}.\boldsymbol{\psi} = \tilde{\mathbf{A}}.\mathbf{G}^{-1}\boldsymbol{\phi} &= \alpha_*\phi + b_*\phi' + c_*\lambda^{-2}\phi'' + d_*\lambda^{-2}\phi''' \text{ (say)} \\ &= \mathbf{a}_*(z,\lambda)\,.\,\boldsymbol{\phi}, \end{aligned}$$

and this is the desired form $\mathcal{N}^*\phi = 0$. We use the formula (2.18 b) to compute $\hat{\mathbf{A}}_0$; the condition $\hat{\mathbf{A}}_0 = 0$ then requires that A, B_0 satisfy

$$-r_0 B_0^1 + z(A_0^1)' + \alpha_* A_0^1 + \beta_* A = 0, \tag{2.24a}$$

$$-q_0 B_0^1 + zA_0^1 + z(B_0^1)' + \alpha_* B_0^1 + \beta_* B_0 = 0, \tag{2.24b}$$

where A_0^1, B_0^1 are given in terms of A, B_0 in (2.21 b); and also (once A, B_0 are known), that C_0^1, D^1 satisfy

$$\begin{aligned} 3(B_0^1)'' - 3z(D^1)'' - 3(1+q_0)\,(D^1)' - (2q_0' + r_0)\,D^1 - p_2(B_0^1 - zD^1) + 3(A_0^1)' - 2z(C_0^1)' \\ -(1+q_0-\alpha_*)\,C_0^1 + \beta_* C = 0, \end{aligned} \tag{2.24c}$$

$$3(B_0^1)' + A_0^1 - 2z(D^1)' - (2+q_0-\alpha_*)\,D^1 + \beta_* z^{-1} B_0 = 0, \tag{2.24d}$$

where C is given in terms of D^1, B_0 in (2.21 b).

To solve (2.24 a, b) for A, B_0, we observe that by formally letting $\lambda \to \infty$ in $\mathscr{M}\psi = 0$ and $\mathscr{N}^*\phi = 0$, we obtain the *reduced equations*

$$\mathscr{R}\psi \equiv z\psi'' + q_0\psi' + r_0\psi = 0, \tag{2.25a}$$

$$\mathscr{R}_0\phi \equiv z\phi'' + \alpha_*\phi' + \beta_*\phi = 0. \tag{2.25b}$$

Formally letting $\lambda \to \infty$ in the transformation $\phi = \mathbf{A}.\boldsymbol{\psi}$, yields $\phi = A\psi + B_0\psi'$, which suggests

LEMMA 2.1. *Let ψ_1, ψ_2 be linearly independent solutions of $\mathscr{R}\psi = 0$ and ϕ_1, ϕ_2 solutions of $\mathscr{R}_0\phi = 0$. If A and B_0 are such that*

$$\phi_j = A\psi_j + B_0\psi_j' \quad (j = 1, 2), \tag{2.26}$$

then A, B_0 satisfy the equations (2.24 a, b) (*but in general are singular at $z = 0$*).

Proof. Differentiate (2.26) and use $\mathscr{R}\psi_j = 0$ to eliminate ψ_j'', thereby obtaining, for $j = 1, 2$,

$$\begin{aligned}\phi_j' &= (A' - r_0 z^{-1} B_0)\,\psi_j + (A + B_0' - q_0 z^{-1} B_0)\,\psi_j' \\ &= A_0^1\psi_j + B_0^1\psi_j'\end{aligned}$$

by (2.21 b). Similarly,

$$\phi_j'' = \{(A_0^1)' - r_0 z^{-1} B_0^1\}\,\psi_j + \{A_0^1 + (B_0^1)' - q_0 z^{-1} B_0^1\}\,\psi_j'.$$

Hence

$$0 = \mathscr{R}_0\phi_j = \{-r_0 B_0^1 + z(A_0^1)' + \alpha_* A_0^1 + \beta_* A\}\,\psi_j + \{-q_0 B_0^1 + zA_0^1 + z(B_0^1)' + \alpha_* B_0^1 + \beta_* B_0\}\,\psi_j', \tag{2.27}$$

and since this is true for linearly indepenent ψ_1, ψ_2, the coefficients of ψ_j and of ψ_j' in (2.27) must both be zero. Now compare (2.27) with the equations (2.24 a, b).

We now show that by correct choice of the constants α_* and β_* we can ensure that A and B_0 belong to $\mathscr{H}(D_2)$, and also satisfy the condition (2.23), and that the matrix $\mathbf{G}$ is then invertible, not only near $z = 0$, but in all of D_2. First, note that the equation $\mathscr{R}\psi = 0$ has a regular singularity at $z = 0$, and, since $q_0(0) = 0$ by (2.3 b), it has indices 1 ($= 1 - q_0(0)$) and 0. Moreover, it is known (from the theory of second-order equations with regular singular points) that $\mathscr{R}\psi = 0$ has solutions ψ_1, ψ_2 defined by

$$\left.\begin{aligned}\psi_1 &= z\left\{1 + \sum_{n=0}^{\infty} c_n z^n\right\}, \\ \psi_2 &= \tilde{\psi}_2 - r_0(0)\ln z\,\psi_1, \\ \text{with}\quad \tilde{\psi}_2 &= 1 + \sum_{n=0}^{\infty} d_n z^n.\end{aligned}\right\} \tag{2.28}$$

Furthermore, the Wronskian of ψ_1, ψ_2 is

$$W_{1,2}(z) \equiv W\{\psi_1, \psi_2\}(z) = \exp\left\{-\int_0^z \frac{q_0(t)}{t}\,\mathrm{d}t\right\} \neq 0 \quad \text{for} \quad z \in D_2, \tag{2.29}$$

since $q_0 \in \mathscr{H}(D_2)$ with $q_0(0) = 0$. Now *choose*

$$\alpha_* = 0, \quad \beta_* = r_0(0); \tag{2.30}$$

for then $\mathscr{R}_0\phi = 0$ *also* has indices 1 ($= 1 - \alpha_*$) and 0, and $\mathscr{R}_0\phi = 0$ has solutions ϕ_1, ϕ_2 defined by

$$\left.\begin{aligned}\phi_1 &= z\left\{1 + \sum_{n=0}^{\infty} \gamma_n z^n\right\}, \\ \phi_2 &= \tilde{\phi}_2 - r_0(0)\ln z\,\phi_1, \\ \text{with}\quad \tilde{\phi}_2 &= 1 + \sum_{n=0}^{\infty} \delta_n z^n.\end{aligned}\right\} \tag{2.31}$$

The choice $\beta_* = r_0(0)$ ensures that the constant $r_0(0)$, which occurs in ψ_2, appears similarly in ϕ_2.

THEOREM 2.2. *Set* $\alpha_* = 0$, $\beta_* = r_0(0)$, *and let* ψ_j *and* $\phi_j (j = 1, 2)$ *be the solutions in* (2.28) *and* (2.31) *of the reduced equations* $\mathscr{R}\psi = 0$ *and* $\mathscr{R}_0\phi = 0$. *Define* $A(z)$ *and* $B_0(z)$ *by the equations*

$$A\psi_j + B_0\psi_j' = \phi_j \quad (j = 1, 2). \tag{2.32}$$

Then A *and* $z^{-1}B_0$ *belong to* $\mathscr{H}(D_2)$ *and satisfy* (2.24*a, b*) *with*

$$A(0) = 1, \quad B_0(z) = O(z^2) \quad \text{for} \quad |z| \to 0; \tag{2.33}$$

and there exist functions $C_0^{\frac{1}{2}}$ *and* D^1 *in* $\mathscr{H}(D_2)$ *that satisfy* (2.24*c, d*). *Moreover, the matrix* $\mathbf{G}$ *in the transformation* $\boldsymbol{\phi} = \mathbf{G}\boldsymbol{\psi}$ *is invertible in* D_2. *This completes the transformation from* $\mathscr{M}\psi = 0$ *to the adjoint normal form* $\mathscr{N}^*\phi = 0$.

Proof. (*a*) The equations (2.32) may be written, because the $\ln z$ terms cancel in the second, as

$$\left.\begin{aligned} Az^{-1}\psi_1 + z^{-1}B_0\psi_1' &= z^{-1}\phi_1 \\ A\tilde{\psi}_2 + z^{-1}B_0(z\tilde{\psi}_2' - r_0(0)\,\psi_1) &= \tilde{\phi}_2, \end{aligned}\right\} \tag{2.34}$$

which may be regarded as a pair of equations for A and $z^{-1}B_0$. Clearly the coefficients and right-hand sides of (2.34) belong to $\mathscr{H}(D_2)$; for the determinant $\Delta_{1,2}$ of the coefficients we have

$$\begin{aligned} \Delta_{1,2} &= \begin{vmatrix} z^{-1}\psi_1 & \psi_1' \\ \tilde{\psi}_2 & z\tilde{\psi}_2' - r_0(0)\,\psi_1 \end{vmatrix} \\ &= \begin{vmatrix} \psi_1 & \psi_1' \\ \tilde{\psi}_2 - r_0(0)\ln z\,\psi_1 & \tilde{\psi}_2' - z^{-1}r_0(0)\,\psi_1 - r_0(0)\ln z\,\psi_1' \end{vmatrix} \\ &= \begin{vmatrix} \psi_1 & \psi_1' \\ \psi_2 & \psi_2' \end{vmatrix} = W_{1,2}(z) \neq 0 \quad \text{for} \quad z \in D_2 \end{aligned}$$

by (2.29). Hence $A, B_0 \in \mathscr{H}(D_2)$. By virtue of lemma 2.1, A and B_0 satisfy the equations (2.24*a, b*). The properties (2.33) of A and B_0 follow if we observe that, at $z = 0$, the pair of equations (2.34) becomes

$$\left.\begin{aligned} A(0).1 + B_0'(0).1 &= 1, \\ A(0).1 \qquad\qquad &= 1, \end{aligned}\right\}$$

whence $A(0) = 1$, $B_0'(0) = 0$.

(*b*) Next, to determine $D^1(z)$, we set $\alpha_* = 0$ in (2.24*d*) and write that equation in the form

$$2z(D^1)' + (2 + q_0)\,D^1 = f_1(z),$$

where $f_1 \in \mathscr{H}(D_2)$. It follows that the function

$$D^1 = \tfrac{1}{2}z^{-1}\exp\left\{-\int_0^z \frac{q_0(t)}{2t}\,\mathrm{d}t\right\}\int_0^z f_1(t)\exp\left\{\int_0^t \frac{q_0(t')}{2t'}\,\mathrm{d}t'\right\}\mathrm{d}t$$

has the properties stated in the theorem. Now set $\alpha_* = 0$ in the equation (2.24*c*) and write it in the form

$$2z(C_0^{\frac{1}{2}})' + (1 + q_0)\,C_0^{\frac{1}{2}} = f_2(z),$$

where $f_2 \in \mathscr{H}(D_2)$. Hence the function

$$C_0^{\frac{1}{2}} = \tfrac{1}{2}z^{-\frac{1}{2}}\exp\left\{-\int_0^z \frac{q_0(t)}{t}\,\mathrm{d}t\right\}\int_0^z t^{-\frac{1}{2}}f_2(t)\exp\left\{\int_0^t \frac{q_0(t')}{2t'}\,\mathrm{d}t'\right\}\mathrm{d}t$$

has the desired properties.

(*c*) The matrix $\boldsymbol{G}$ in the transformation $\boldsymbol{\phi} = \boldsymbol{G}\boldsymbol{\psi}$ is now known explicitly in terms of the functions $A(z)$, $B_0(z)$, $C_0^1(z)$, $D^1(z)$, and the coefficients $p(z,\lambda)$, $q(z,\lambda)$, $r(z,\lambda)$ of $\mathscr{M}\psi = 0$. Moreover, $\boldsymbol{G} \in \Lambda$, and it remains to show that $\boldsymbol{G}$ is invertible in D_2. First, recall that

$$B_0^1 = B_0' + A - q_0 z^{-1} B_0$$

by (2.21 *b*), so that the determinations $A(0) = 1$, $B_0'(0) = 0$ in (*a*) above imply that $B_0^1(0) = 1$. Therefore, by (2.22),

$$(\det \boldsymbol{G}_0)(0) = A(0)\{B_0^1(0)\}^3 = 1. \tag{2.35}$$

Next, observe that if $\boldsymbol{\Psi}$ is a fundamental matrix solution of the system (2.4): $\boldsymbol{\psi}' = \boldsymbol{M}\boldsymbol{\psi}$, then the matrix $\boldsymbol{\Phi} = \boldsymbol{G}\boldsymbol{\Psi}$ is a fundamental matrix solution of the system (2.7): $\boldsymbol{\phi}' = \boldsymbol{N}^*\boldsymbol{\phi}$. But the equation $\mathscr{M}\psi = 0$ does not contain ψ''', and the coefficient of ϕ''' in $\mathscr{N}^*\phi = 0$ is $O(\lambda^{-2})$ for $|\lambda| \to \infty$. It follows that

$$\det \boldsymbol{G} = (\det \boldsymbol{\Phi})(\det \boldsymbol{\Psi})^{-1} = \text{constant} + O(\lambda^{-2}).$$

Hence,

$$\det \boldsymbol{G}_0(z) = 1 \quad \text{in} \quad D_2 \tag{2.36}$$

by (2.35), from which it is clear that the matrix $\boldsymbol{G}$ is invertible in D_2 for large $|\lambda|$.

(*c*) *Formal, approximate solutions of* $\mathscr{N}\chi = 0$

In order to make the theory of Lin & Rabenstein (1969) applicable, it is necessary† to consider the normal form

$$\mathscr{N}\chi \equiv \chi^{\text{iv}} + \lambda^2\{z\chi'' + \alpha_0\chi' + \beta_0\chi\} - \boldsymbol{a}(z,\lambda).\boldsymbol{\chi} = 0 \quad \text{in} \quad D_2, \tag{2.37}$$

where $\mathscr{N}$ and $\mathscr{N}^*$ are (real-type) adjoints of each other, so that $\alpha_0 = 2 - \alpha_* = 2$, $\beta_0 = \beta_* = r_0(0)$, and $\boldsymbol{a}(z,\lambda) = (a,b,c,d) \in \Lambda$ is given in terms of $\boldsymbol{a}_*(z,\lambda) = (a_*, b_*, c_*, d_*) \in \Lambda$ in (2.9). *Henceforth, although we shall assume that*

$$\alpha_0 = 2, \quad \beta_0 = r_0(0), \tag{2.38}$$

in $\mathscr{N}\chi = 0$, *we nevertheless retain the symbols* α_0 *and* β_0. Now introduce the dominant differential operator L_0 defined by

$$L_0 u \equiv u^{\text{iv}} + \lambda^2\{zu'' + \alpha_0 u' + \beta_0 u\},$$

and note that $\mathscr{N}\chi = 0$ can also be written in the form

$$L_0\chi - \boldsymbol{a}(z,\lambda).\boldsymbol{\chi} = 0 \quad \text{in} \quad D_2.$$

Next, we define the basic reference equation

$$\mathscr{Q}u \equiv u^{\text{iv}} + \lambda^2\{zu'' + \alpha(\lambda)\,u' + \beta(\lambda)\,u\} = 0 \quad \text{in} \quad D_2, \tag{2.39}$$

or, equivalently, in vector form, $\boldsymbol{u}' = \boldsymbol{Q}\boldsymbol{u}$ as in (2.13). Here the coefficients α and β are in $\Lambda(2m)$ for some non-negative integer m, and are independent of z; they are undetermined apart from $\alpha_0 = 2$ and $\beta_0 = r_0(0)$.

We now construct a formally approximate solution $\chi_{(2m)}$ of $\mathscr{N}\chi = 0$. Let u be a given solution of $\mathscr{Q}u = 0$, even though α_{2n}, β_{2n} $(n = 1, 2, \ldots, m)$ are still to be chosen, and m an arbitrary non-negative integer. Make the transformation

$$\chi_{(2m)} = \boldsymbol{\mu}(z,\lambda).\boldsymbol{u} = \mu u + \nu u' + \sigma\lambda^{-2}u'' + \tau\lambda^{-2}u''', \tag{2.40}$$

† This is due to the restriction Re $(\alpha_0) > 1$ imposed by Lin & Rabenstein (1969).

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where $\boldsymbol{\mu}(z,\lambda) = (\mu, \nu, \sigma, \tau)$ is a row vector in $\Lambda(2m+2)$ to be chosen, with $\mu, \sigma, \tau \in \Lambda(2m)$, and $\nu \in \Lambda(2m+2)$. Equivalently (cf. (2.18), (2.19)),

$$\left.\begin{aligned} \boldsymbol{\chi}_{(2m)} &= \boldsymbol{H}_{(2m)}\boldsymbol{u}, \\ \text{where} \qquad \boldsymbol{H}_{(2m)} &= \begin{bmatrix} \mu & \nu & \sigma & \tau \\ \mu^1 & \nu^1 & \sigma^1 & \tau^1 \\ \lambda^{-2}\mu^2 & \lambda^{-2}\nu^2 & \lambda^{-2}\sigma^2 & \lambda^{-2}\tau^2 \\ \lambda^{-2}\mu^3 & \lambda^{-2}\nu^3 & \lambda^{-2}\sigma^3 & \lambda^{-2}\tau^2 \end{bmatrix}, \end{aligned}\right\} \tag{2.41}$$

$$\text{and} \qquad \boldsymbol{\mu}^0 = \boldsymbol{\mu}, \quad \boldsymbol{\mu}^{j+1} = (\boldsymbol{\mu}^j)' + \boldsymbol{\mu}^j \boldsymbol{Q} \quad (j = 0, 1, 2, 3). \tag{2.42}$$

The matrix $\boldsymbol{Q}$ is given in (2.13). It is clear, if we replace B_0 and D by ν_0 and τ_0 in (2.20), that

$$\boldsymbol{H}_{(2m)} \in \Lambda(4m+6) \quad \text{if} \quad (\text{and only if}) \quad \nu_0 = z\tau_0. \tag{2.43}$$

Here the argument $4m+6$ is obtained by first noting that the functions

$$\mu, \sigma, \tau \in \Lambda(2m), \quad \nu \in \Lambda(2m+2),$$

and the coefficients $\alpha, \beta \in \Lambda(2m)$, and then using the formula (2.42) for $j = 0, 1, 2, 3$. With $\chi_{(2m)}$ as in (2.41), we have

$$\begin{aligned} \mathscr{N}\chi_{(2m)} &= \boldsymbol{\mu}^4.\boldsymbol{u} + \lambda^2\{z\boldsymbol{\mu}^2.\boldsymbol{u} + \alpha_0\boldsymbol{\mu}^1.\boldsymbol{u} + \beta_0\boldsymbol{\mu}.\boldsymbol{u}\} - \{a\boldsymbol{\mu}.\boldsymbol{u} + b\boldsymbol{\mu}^1.\boldsymbol{u} + c\lambda^{-2}\boldsymbol{\mu}^2.\boldsymbol{u} + d\lambda^{-2}\boldsymbol{\mu}^3.\boldsymbol{u}\} \\ &\equiv \lambda^2\hat{\boldsymbol{\mu}}.\boldsymbol{u} \quad (\text{say}), \end{aligned}$$

with $\hat{\boldsymbol{\mu}} \in \Lambda$, as can be verified from the formula (2.42). Assume, for the moment, that the matrix $\boldsymbol{H}_{(2m)}$ in the transformation $\boldsymbol{\chi}_{(2m)} = \boldsymbol{H}_{(2m)}\boldsymbol{u}$ is invertible, and that

$$\hat{\boldsymbol{\mu}} = \lambda^{-2m-2}\tilde{\boldsymbol{\mu}}, \quad \text{with} \quad \tilde{\boldsymbol{\mu}} \in \Lambda. \tag{2.44}$$

In that case,

$$\begin{aligned} \mathscr{N}\chi_{(2m)} &= \lambda^{-2m}\tilde{\boldsymbol{\mu}}.\boldsymbol{u} = \lambda^{-2m}\tilde{\boldsymbol{\mu}}.\boldsymbol{H}^{-1}_{(2m)}\boldsymbol{\chi}_{(2m)} \\ &= \lambda^{-2m}\hat{\boldsymbol{a}}(z,\lambda).\boldsymbol{\chi}_{(2m)}, \quad \text{say}, \end{aligned}$$

with $\hat{\boldsymbol{a}}(z,\lambda) = (\hat{a}, \hat{b}, \hat{c}, \hat{d}) \in \Lambda$, and this is the desired related equation (2.16). To realize (2.44), we must have

$$\hat{\boldsymbol{\mu}}_{2n} = 0 \quad (n = 0, 1, \ldots, m). \tag{2.45}$$

Note that what follows resembles the work of § 2 (*b*), with $\boldsymbol{\mu}$, $\hat{\boldsymbol{\mu}}$ replacing $\boldsymbol{A}$, $\hat{\boldsymbol{A}}$. By (2.42), and the fact that $\nu_0 = z\tau_0$, condition (2.45) for $n = 0$ requires that μ_0, τ_0 satisfy

$$z\mu_0'' + \alpha_0\mu_0' - \beta_0(2z\tau_0' + \tau_0) = 0, \tag{2.46a}$$

$$z\tau_0'' + (2-\alpha_0)\,\tau_0' + 2\mu_0' = 0, \tag{2.46b}$$

and, once μ_0, τ_0 are known, that σ_0^1, τ_0^1 satisfy

$$3(\nu_0^1)'' - 3z(\tau_0^1)'' - 3(1+\alpha_0)\,(\tau_0^1)' - \beta_0\tau_0' + 3(\mu_0^1)' - 2z(\sigma_0^1)' - \sigma_0^1 = 0, \tag{2.46c}$$

$$3(\nu_0^1)' + \mu_0^1 - 2z(\tau_0^1)' - 2\tau_0^1 + \beta_0\tau_0 = 0, \tag{2.46d}$$

$$\begin{aligned} \text{where} \qquad \mu_0^1 &= \mu_0' - \beta_0\tau_0, \\ \nu_0^1 &= (z\tau_0)' + \mu_0 - \alpha_0\tau_0. \end{aligned}$$

After σ_0^1, τ_0^1 have been found, the relations

$$\begin{aligned} \sigma_0^1 &= \sigma_0' + \nu_2 - z\tau_2, \\ \tau_0^1 &= \tau_0' + \sigma_0, \end{aligned}$$

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and the formula (2.42) for $\boldsymbol{\mu}^j$ $(j = 2, 3)$ complete the construction of the matrix $(\boldsymbol{H}_{(2m)})_0$. Once matrices $(\boldsymbol{H}_{(2m)})_0, (\boldsymbol{H}_{(2m)})_2, \ldots, (\boldsymbol{H}_{(2m)})_{2n-2} \in \mathscr{H}(D_2)$ have been determined, condition (2.45) can be shown to require, for $n = 1, 2, \ldots, m$, that the functions μ_{2n}, τ_{2n} satisfy

$$z\mu''_{2n} + \alpha_0 \mu'_{2n} - \beta_0(2z\tau'_{2n} + \tau_{2n}) = \beta_{2n} + F_{2n}(z), \tag{2.47a}$$

$$z\{z\tau''_{2n} + (2-\alpha_0)\,\tau'_{2n} + 2\mu'_{2n}\} = \alpha_{2n} + G_{2n}(z), \tag{2.47b}$$

where F_{2n} and G_{2n} are *known* functions in $\mathscr{H}(D_2)$ *provided* that (*a*) the constants α_{2j}, β_{2j} $(j = 1, \ldots, 2n-2)$ have been determined, and (*b*) the leading matrix $(\boldsymbol{H}_{(2m)})_0 = I$; and where we have used the fact that

$$\nu_{2n} = z\tau_{2n} + \sigma^1_{2n-2} - \sigma'_{2n-2}.$$

After functions $\mu_{2n}, \tau_{2n} \in \mathscr{H}(D_2)$ and the constants α_{2n}, β_{2n} have been found, the functions $\sigma^1_{2n}, \tau^1_{2n}$ must satisfy

$$-3z(\tau^1_{2n})'' - 3(1+\alpha_0)\,(\tau^1_{2n})' - 2z(\sigma^1_{2n})' - \sigma^1_{2n} = f_{2n}(z), \tag{2.47c}$$

$$z(\tau^1_{2n})' + \tau^1_{2n} = g_{2n}(z), \tag{2.47d}$$

where f_{2n} and g_{2n} are known functions in $\mathscr{H}(D_2)$. As before, determination of the functions $\sigma^1_{2n}, \tau^1_{2n} \in \mathscr{H}(D_2)$ completes the construction of $(\boldsymbol{H}_{(2m)})_{2n}$, and this matrix then belongs to $\mathscr{H}(D_2)$. In fact, once $(\boldsymbol{H}_{(2m)})_{2m}$ has been determined, it is clear that the matrix $\boldsymbol{H}_{(2m)} \in \Lambda(4m+6)$ will be known.

Let m be an arbitrary non-negative integer. With $\alpha_0 = 2$, $\beta_0 = r_0(0)$, we can now prove

THEOREM 2.3. *One can find (in principle)* (i) *coefficients* $\alpha, \beta \in \Lambda(2m)$ *for the basic reference equation* $\mathscr{Q}u = 0$; (ii) *a matrix* $\boldsymbol{H}_{(2m)} \in \Lambda(4m+6)$, *with*

$$(\boldsymbol{H}_{(2m)})_0 = I, \tag{2.48}$$

such that a given solution u of $\mathscr{Q}u = 0$ *generates, by means of the transformation*

$$\boldsymbol{\chi}_{(2m)}(z, \lambda) = \boldsymbol{H}_{(2m)}(z, \lambda)\,\boldsymbol{u}(z, \lambda), \tag{2.49}$$

a formally approximate solution $\chi_{(2m)}$ *of the normal form* $\mathscr{N}\chi = 0$. *More precisely,* $\chi_{(2m)}$ *satisfies a related equation of the form*

$$\mathscr{N}\chi_{(2m)} = \lambda^{-2m}\hat{\boldsymbol{a}}(z, \lambda)\,.\,\boldsymbol{\chi}_{(2m)},$$

where $\hat{\boldsymbol{a}}(z, \lambda) = (\hat{a}, \hat{b}, \hat{c}, \hat{d}) \in \Lambda$. *(If (and only if)* $\beta_0 \neq 0$, *the constants* β_{2n} $(n = 1, 2, \ldots, m)$ *may be assigned arbitrary values).*

Proof. (*a*) Introduce the vector $\boldsymbol{\chi}_{(2m)} = \boldsymbol{H}_{(2m)}\boldsymbol{u}$ as in (2.41), and set $\alpha_0 = 2$ in the systems of equations (2.46) and (2.47). It is clear that if we set

$$\mu_0 = 1, \quad \nu_0 = 0, \quad \sigma^1_0 = 0, \quad \tau^1_0 = 0,$$

the system (2.46) is satisfied. From (2.42) we then deduce that $(\boldsymbol{H}_{(2m)})_0 = I$.

(*b*) Let h be a generic symbol for functions in $\mathscr{H}(D_2)$. Now consider (2.47*b*). We set the constant

$$\alpha_{2n} = -G_{2n}(0),$$

multiply the equation by z^{-1}, and integrate the resulting equation once to obtain

$$z\tau'_{2n} - \tau_{2n} + 2\mu_{2n} = h(z).$$

Now use this equation to eliminate $z\tau'_{2n}$ from (2.47 a). The pair of equations (2.47 a, b) consequently becomes

$$z\mu''_{2n} + 4\beta_0 \mu_{2n} + 2\mu'_{2n} - 3\beta_0 \tau_{2n} = \beta_{2n} + h(z), \tag{2.50a}$$

$$z\tau'_{2n} - \tau_{2n} + 2\mu_{2n} = h(z), \tag{2.50b}$$

or, equivalently, in vector form,

$$\boldsymbol{w}' = \boldsymbol{A}(z)\,\boldsymbol{w} + \boldsymbol{f}(z), \tag{2.50c}$$

where $$w = \begin{bmatrix} \mu_{2n} \\ \mu'_{2n} \\ \tau_{2n} \end{bmatrix}, \quad A(z) = \begin{bmatrix} 0 & 1 & 0 \\ -\dfrac{4\beta_0}{z} & -\dfrac{2}{z} & \dfrac{3\beta_0}{z} \\ -\dfrac{2}{z} & 0 & \dfrac{1}{z} \end{bmatrix}, \quad f(z) = \begin{bmatrix} 0 \\ \dfrac{\beta_{2n} + h(z)}{z} \\ \dfrac{h(z)}{z} \end{bmatrix}.$$

Since the system $\boldsymbol{w}' = \boldsymbol{A}(z)\,\boldsymbol{w}$ has a singularity of the first kind at $z = 0$, it follows from the theory of such systems (Coddington & Levinson 1955, chapter 4) that formal power series solutions of the system (2.50 c), and therefore of the pair of equations (2.50 a, b), are in fact strict solutions. Accordingly we set

$$\mu_{2n} = \sum_{k=0}^{\infty} \mu_{n,k} z^k, \quad \tau_{2n} = \sum_{k=0}^{\infty} \tau_{n,k} z^k, \quad h = \sum_{k=0}^{\infty} h_k z^k.$$

Substituting these series into the equations (2.50 a, b) then leads to

$$(k+1)(k+2)\,\mu_{n,k+1} + 4\beta_0 \mu_{n,k} - 3\beta_0 \tau_{n}, k = \beta_{2n}\,\delta_{k,0} + h_k \quad (k = 0, 1, \ldots), \tag{2.51a}$$

$$(k-1)\,\tau_{n,k} + 2\mu_{n,k} = h_k \quad (k = 0, 1, \ldots), \tag{2.51b}$$

where $\delta_{k,0}$ denotes the Kronecker delta. It is clear that the equation (2.51 b) can be solved for $\mu_{n,k}$ once $\tau_{n,k}$ is known. To determine the functions $\tau_{n,k}$, we eliminate $\mu_{n,k+1}$ and $\mu_{n,k}$ from the equation (2.51 a) by means of (2.51 b), thereby obtaining, for $k = 0, 1, \ldots,$ the system

$$2\beta_0(2k+1)\,\tau_{n,k} + k(k+1)(k+2)\,\tau_{n,k+1} = -2\beta_{2n}\,\delta_{k,0} + c_{n,k}, \tag{2.52}$$

where $c_{n,k}(k = 0, 1, \ldots)$ are known constants. If $\beta_0 \neq 0$, we can, for arbitrary β_{2n}, solve this system for $\tau_{n,k}\,(k = 1, 2, \ldots)$ in terms of the leading coefficient $\tau_{n,0}$. If $\beta_0 = 0$, we choose

$$\beta_{2n} = \tfrac{1}{2}c_{n,0},$$

to obtain a one-parameter family of solutions $\tau_{n,k}\,(k = 1, 2, \ldots)$ of the system (2.52). Hence, by correct choice of the constants α_{2n}, β_{2n}, we have found functions μ_{2n}, $\tau_{2n} \in \mathscr{H}(D_2)$ satisfying equations (2.47 a, b); and so, from equation (2.47 c), we obtain

$$\tau^1_{2n} = z^{-1}\int_0^z g_{2n}(t)\,\mathrm{d}t \in \mathscr{H}(D_2),$$

and then, by equation (2.47 d), we have

$$\sigma^1_{2n} = \tfrac{1}{2}z^{-\frac{1}{2}}\int_0^z t^{-\frac{1}{2}}h(t)\,\mathrm{d}t \in \mathscr{H}(D_2).$$

(c) Finally, since $(\boldsymbol{H}_{2m})_0 = I$, the matrix $\boldsymbol{H}_{(2m)}$ is invertible in D_2 for large $|\lambda|$. The arguments preceding the theorem are now relevant, and the result follows.

(d) Strict solutions of $\mathscr{N}\chi = 0$

By virtue of theorem 2.3 we can now appeal to the theory of Lin & Rabenstein (1969) to justify the validity of the formally approximate solution $\chi_{(2m)}$ of $\mathscr{N}\chi = 0$ constructed above.

With a view to the applications in §§4–5, we now specialize to the following situation. By a *fixed* number or domain (open, connected, non-empty set) we mean one independent of z and λ.

(*a*) *λ is restricted to the sector*

$$|\arg\lambda + \tfrac{1}{4}\pi| < \delta \tag{2.53}$$

for some fixed (small) $\delta > 0$.

(*b*) *The domain* D_2 (which is characterized by the fact that the coefficients p, q, r of $\mathscr{M}\psi = 0$ are holomorphic functions of z there) *is a fixed neighbourhood of a given fixed closed real interval* $[z_a, z_b]$, *with* $z_a < 0 < z_b$.

Let us define the disk

$$\Omega = \{z \mid |z| < K|\lambda|^{-\frac{2}{3}}\} \tag{2.54}$$

for some fixed (large) positive K; then $\Omega \subset D_2$ for $|\lambda|$ sufficiently large.

In order to define paths of integration later, and because D_2 need *not* contain a disk about the origin of radius $\max\{-z_a, z_b\}$, we also introduce the sets (figure 1)

$$\left.\begin{aligned} S'_1 &= \{z \mid |\arg z| < \delta_1, \quad |z| < R'_1, \quad z \notin \Omega\},\\ S'_2 &= \{z \mid |\arg z + \pi| < \delta_1, \quad |z| < R'_2, \quad z \notin \Omega\},\\ S'_3 &= \{z \mid -\pi + \delta_1 < \arg z < -\delta_1, \quad |z| < R'_3, \quad z \notin \Omega\},\end{aligned}\right\} \tag{2.55}$$

where δ_1 and R'_j $(j = 1, 2, 3)$ are fixed positive numbers such that each $\overline{S'_j} \subset D_2$, with $z_a \in S'_2$, $z_b \in S'_1$ and $2R'_3 \leqslant \min\{R'_1, R'_2\}$. We define

$$S' = S'_1 \cup S'_2 \cup S'_3, \quad \overline{D'_3} = \overline{\Omega} \cup \overline{S'}, \quad D'_3 = \operatorname{int} \overline{D'_3}, \tag{2.56}$$

and add the restriction

$$\delta + \tfrac{3}{2}\delta_1 < \tfrac{1}{4}\pi, \tag{2.57}$$

(because we shall need the condition

$$-\tfrac{3}{2}\pi < \arg(\mathrm{i}\lambda z^{\frac{3}{2}}) < \tfrac{1}{2}\pi \quad \text{when} \quad z \in \overline{D'_3}). \tag{2.58}$$

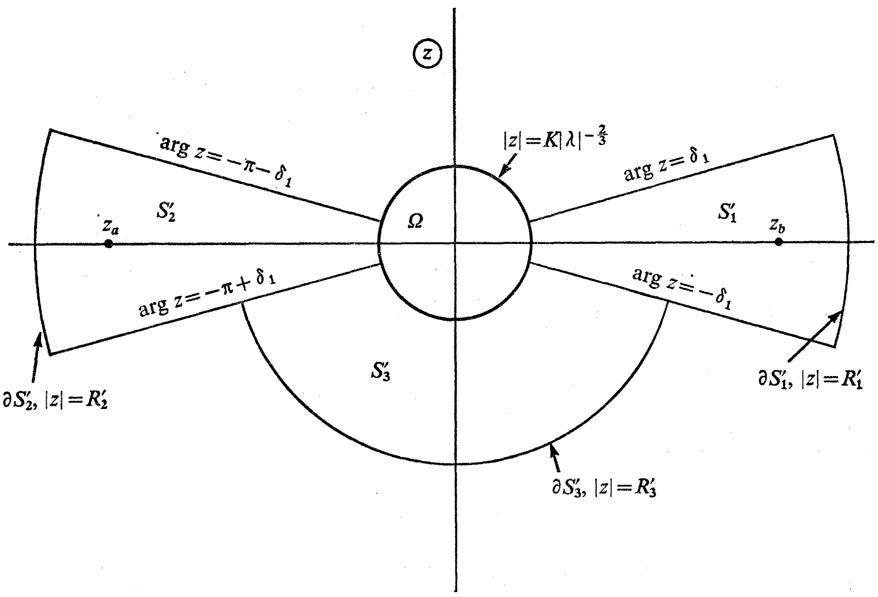

FIGURE 1. The domain D'_3 in the z-plane as defined by (2.56). The points z_a and z_b are also shown.

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Next, we recall that the basic reference equation $\mathcal{Q}u = 0$ is equivalent to the first order system (2.13): $\boldsymbol{u}' = \boldsymbol{Q}\boldsymbol{u}$, and that the coefficients $\alpha, \beta \in A(2m)$ of $\mathcal{Q}u = 0$ are chosen as in theorem 2.3. *We assume henceforth that* $\beta_0 \neq 0$; we can then, according to theorem 2.3, choose the constants β_{2n} $(n = 1, 2, \ldots, m)$ all equal to zero, and it follows that $\beta = \beta_0$.

Now introduce the fundamental set of solutions

$$B_0(z;\alpha(\lambda),\beta_0,\lambda),\quad A_1(z;\alpha(\lambda),\beta_0,\lambda),\quad A_2(z;\alpha(\lambda),\beta_0,\lambda),\quad B_3(z;\alpha(\lambda),\beta_0,\lambda)$$

of $\mathcal{Q}u = 0$ (as given in appendix A). Then the matrix

$$\boldsymbol{U}(z;\alpha(\lambda),\beta_0,\lambda) = \begin{bmatrix} B_0 & A_1 & A_2 & B_3 \\ B_0' & A_1' & A_2' & B_3' \\ \lambda^{-2}B_0'' & \lambda^{-2}A_1'' & \lambda^{-2}A_2'' & \lambda^{-2}B_3'' \\ \lambda^{-2}B_0''' & \lambda^{-2}A_1''' & \lambda^{-2}A_2''' & \lambda^{-2}B_3''' \end{bmatrix} \tag{2.59}$$

is a fundamental matrix solution of $\boldsymbol{u}' = \boldsymbol{Q}\boldsymbol{u}$.

Let $E(z,\lambda)$ be a generic symbol for functions which are continuous and uniformly bounded for $z \in D_3'$ and $|\lambda| \geqslant |\lambda_0|$, with $|\lambda_0|$ sufficiently large, and denote by $\boldsymbol{E}(z,\lambda)$ a matrix of the form

$$\boldsymbol{E}(z,\lambda) = \begin{bmatrix} E(z,\lambda) & E(z,\lambda) & E(z,\lambda) & E(z,\lambda) \\ E(z,\lambda) & E(z,\lambda) & E(z,\lambda) & E(z,\lambda) \\ E(z,\lambda) & E(z,\lambda) & E(z,\lambda) & E(z,\lambda) \\ E(z,\lambda) & E(z,\lambda) & E(z,\lambda) & E(z,\lambda) \end{bmatrix}.$$

Furthermore, introduce the matrix operation $\times$ as follows. If $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ are two given $n \times n$ matrices, then the matrix $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$, with $\boldsymbol{C} = (c_{ij})$, is defined by $c_{ij} = a_{ij}b_{ij}$.

Recalling that the normal form $\mathcal{N}\chi = 0$ is equivalent to the system (2.8): $\boldsymbol{\chi}' = \boldsymbol{N}\boldsymbol{\chi}$, we can now prove the following theorem.

THEOREM 2.4. *Suppose that the matrix* $\boldsymbol{H}_{(2m)}$ *in the transformation* $\boldsymbol{\chi}_{(2m)} = \boldsymbol{H}_{(2m)}\boldsymbol{u}$ *is chosen as in theorem* 2.3. *Let* $\boldsymbol{U}(z;\alpha(\lambda),\beta_0,\lambda)$ *be the fundamental matrix solution* (2.59) *of* $\boldsymbol{u}' = \boldsymbol{Q}\boldsymbol{u}$, *and define*

$$\xi = \tfrac{2}{3}\mathrm{i}\lambda z^{\frac{3}{2}}. \tag{2.60}$$

Then the system $\boldsymbol{\chi}' = \boldsymbol{N}\boldsymbol{\chi}$ *has, for* $z \in D_3'$ *and* $|\lambda|$ *sufficiently large, a fundamental matrix solution*

$$\boldsymbol{X}(z,\lambda) = \boldsymbol{H}_{(2m)}\boldsymbol{U} + \lambda^{-2m-2}\boldsymbol{E}(z,\lambda) \times \boldsymbol{R}(z,\lambda), \tag{2.61}$$

where

$$\boldsymbol{R} = \begin{bmatrix} 1 & \lambda^{\frac{2}{3}}\ln\lambda & \lambda^{\frac{2}{3}}\ln\lambda & \lambda^{\frac{2}{3}}\ln\lambda \\ 1 & \lambda^{\frac{4}{3}}\ln\lambda & \lambda^{\frac{4}{3}}\ln\lambda & \lambda^{\frac{4}{3}}\ln\lambda \\ \lambda^{-2} & \ln\lambda & \ln\lambda & \ln\lambda \\ \lambda^{-2} & \lambda^{\frac{2}{3}}\ln\lambda & \lambda^{\frac{2}{3}}\ln\lambda & \lambda^{\frac{2}{3}}\ln\lambda \end{bmatrix} \quad \textit{for} \quad z \in \Omega, \tag{2.62a}$$

$$\boldsymbol{R} = \begin{bmatrix} 1 & \lambda^{\frac{1}{2}}\ln\lambda\, z^{-\frac{1}{4}}\mathrm{e}^{-\xi} & \lambda^{\frac{1}{2}}\ln\lambda\, z^{-\frac{1}{4}}\mathrm{e}^{\xi} & \ln\lambda\, z^{-1} \\ 1 & \lambda^{\frac{3}{2}}\ln\lambda\, z^{\frac{1}{4}}\mathrm{e}^{-\xi} & \lambda^{\frac{3}{2}}\ln\lambda\, z^{\frac{1}{4}}\mathrm{e}^{\xi} & \ln\lambda\, z^{-2} \\ \lambda^{-2} & \lambda^{\frac{1}{2}}\ln\lambda\, z^{\frac{3}{4}}\mathrm{e}^{-\xi} & \lambda^{\frac{1}{2}}\ln\lambda\, z^{\frac{3}{4}}\mathrm{e}^{\xi} & \lambda^{-2}\ln\lambda\, z^{-3} \\ \lambda^{-2} & \lambda^{\frac{3}{2}}\ln\lambda\, z^{\frac{5}{4}}\mathrm{e}^{-\xi} & \lambda^{\frac{3}{2}}\ln\lambda\, z^{\frac{5}{4}}\mathrm{e}^{\xi} & \lambda^{-2}\ln\lambda\, z^{-4} \end{bmatrix} \quad \textit{for} \quad z \in D_3'\backslash\Omega. \tag{2.62b}$$

Moreover,

$$\det \boldsymbol{X} \sim 4\pi^2\beta_0\lambda^2 \quad \textit{for} \quad |\lambda| \to \infty \quad \textit{and} \quad z \in D_3'. \tag{2.63}$$

Proof. We follow Lin & Rabenstein (1969), except that the lemmas bounding certain integrals in § 6 of that paper must now be replaced by those in § 3 (*b*) below. In the definition (2.62) of the matrix $\boldsymbol{R}$, and the asymptotic property (2.63) of $\det \boldsymbol{X}$, we have used the fact that $\alpha_0 = 2$.

(e) *Asymptotic solutions of* $\mathscr{N}^*\phi = 0$

In § 2 (*b*), we transformed the equation $\mathscr{M}\psi = 0$ to the *adjoint normal form*

$$\mathscr{N}^*\phi \equiv \phi^{\text{iv}} + \lambda^2\{z\phi'' + \beta_0\phi\} - \boldsymbol{a}_*(z, \lambda) \,.\, \boldsymbol{\phi} = 0,$$

with $\boldsymbol{a}_*(z,\lambda) = (a_*, b_*, c_*, d_*) \in \Lambda$. Theorem 2.4 now enables us to obtain a fundamental set of solutions, with given asymptotic properties for $|\lambda| \to \infty$, of $\mathscr{N}^*\phi = 0$.

Theorem 2.5. *For* $z \in D_3'$ *and* $|\lambda|$ *sufficiently large, the equation* $\mathscr{N}^*\phi = 0$ *has a fundamental set of solutions*

$$\phi_0 \sim 2\pi \mathrm{i} z^{\frac{1}{2}} \beta_0^{-\frac{1}{2}} J_1(2z^{\frac{1}{2}}\beta_0^{\frac{1}{2}}) \quad \textit{for} \quad z \in D_3', \tag{2.64a}$$

$$\left.\begin{aligned} \phi_1 &\sim -\mathrm{i}\sqrt{\pi}\, \mathrm{e}^{-\frac{3}{4}\pi\mathrm{i}} \lambda^{-\frac{3}{2}} z^{-\frac{5}{4}} \mathrm{e}^{-\xi}, \\ \phi_2 &\sim \sqrt{\pi}\, \mathrm{e}^{-\frac{3}{4}\pi\mathrm{i}} \lambda^{-\frac{3}{2}} z^{-\frac{5}{4}} \mathrm{e}^{\xi}, \\ \phi_3 &\sim \pi \mathrm{i} z^{\frac{1}{2}} \beta_0^{-\frac{1}{2}} H_1^{(1)}(2z^{\frac{1}{2}}\beta_0^{\frac{1}{2}}) \end{aligned}\right\} \quad \textit{for fixed} \quad z \in D_3' \backslash \{0\}, \tag{2.64b}$$

(*where* J_1 *and* $H_1^{(1)}$ *are Bessel functions in the standard notation*), *with the following detailed properties.*

(i) *The functions* $(\mathrm{d}/\mathrm{d}z)^j\phi_k$ $(j = 0, 1, 2, 3; \quad k = 0, 1, 2, 3)$ *have the following bounds in* D_3':

$$\left.\begin{aligned} &\phi_0 = zE(z,\lambda), \quad (\mathrm{d}/\mathrm{d}z)^j\phi_0 = E(z,\lambda) \quad (j = 1,2,3); \\ &(\mathrm{d}/\mathrm{d}z)^j\phi_1 = \begin{cases} \lambda^{-\frac{2}{3}(1-j)}E(z,\lambda) \quad (j = 0,1,2,3) \quad \textit{for} \quad z \in \Omega, \\ \lambda^{j-\frac{3}{2}} z^{\frac{1}{2}j-\frac{5}{4}} \mathrm{e}^{-\xi} E(z,\lambda) \quad (j = 0,1,2,3) \quad \textit{for} \quad z \in D_3' \backslash \Omega; \end{cases} \\ &(\mathrm{d}/\mathrm{d}z)^j\phi_2 = \begin{cases} \lambda^{-\frac{2}{3}(1-j)}E(z,\lambda) \quad (j = 0,1,2,3) \quad \textit{for} \quad z \in \Omega, \\ \lambda^{j-\frac{3}{2}} z^{\frac{1}{2}j-\frac{5}{4}} \mathrm{e}^{\xi} E(z,\lambda) \quad (j = 0,1,2,3) \quad \textit{for} \quad z \in D_3' \backslash \Omega; \end{cases} \\ &\phi_3 = E(z,\lambda), \\ &\begin{cases} (\mathrm{d}\phi_3/\mathrm{d}z) = \ln\lambda\, E(z,\lambda), \quad (\mathrm{d}/\mathrm{d}z)^j\phi_3 = \lambda^{-\frac{2}{3}(1-j)}E(z,\lambda) \quad (j = 2,3) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for} \quad z \in \Omega, \\ (\mathrm{d}/\mathrm{d}z)^j\phi_3 = \{(\mathrm{d}/\mathrm{d}z)^{j-1}\ln z\}\, E(z,\lambda) \quad (j = 1,2,3) \quad \textit{for} \quad z \in D_3' \backslash \Omega. \end{cases} \end{aligned}\right\} \tag{2.65}$$

(ii) *Suppose that* $z_* \in D_3'$, *with* $|z_*| \geqslant \delta > 0$ *for some fixed number* δ. *Then, in a sufficiently small fixed neighbourhood* N_* *of* z_*, *we have for* ϕ_0 *and* ϕ_3 *that*

$$\phi_k = \sum_{n=0}^{m} \lambda^{-2n}\phi_{k,2n}(z) + R_{k,m}(z,\lambda) \quad (k = 0, 3), \tag{2.66a}$$

where the functions $\phi_{k,2n}$ $(k = 0, 3; n = 0, 1, \ldots, m)$ *all belong to* $\mathscr{H}(N_*)$, *where*

$$(\mathrm{d}/\mathrm{d}z)^j R_{k,m}(z,\lambda) = \lambda^{-2m-2}\ln\lambda\, E(z,\lambda) \quad (k = 0, 3; j = 0, \ldots, 4), \tag{2.66b}$$

and where m *is an arbitrary non-negative integer.*

(iii) *The Wronskian*

$$W(\phi_0, \phi_1, \phi_2, \phi_3) \sim \frac{\mathrm{i}}{2\pi}\beta_0^{-1}\lambda^2 \quad \textit{for} \quad |\lambda| \to \infty. \tag{2.67}$$

Proof. (*a*) First, recall that the normal form $\mathscr{N}\chi = 0$ and the adjoint normal form $\mathscr{N}^*\phi = 0$ can be written $\boldsymbol{\chi}' = \boldsymbol{N}\boldsymbol{\chi}$ and $\boldsymbol{\phi}' = \boldsymbol{N}^*\boldsymbol{\phi}$ (respectively) as in (2.8) and (2.7). Next, denoting by $\boldsymbol{A}^T = (a_{ji})$ the *transpose* of a matrix $\boldsymbol{A} = (a_{ij})$, we observe that vector solutions

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \quad \text{of} \quad \boldsymbol{v}' = -(\boldsymbol{N}^*)^T\boldsymbol{v}$$

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are related to scalar solutions χ of $\mathcal{N}\chi = 0$ by

$$\left.\begin{aligned} v_1 &= -\lambda^{-2}\chi''' - z\chi' - \chi + \lambda^{-2}(-b_* + \lambda^{-2}c_*' - \lambda^{-2}d_*'')\,\chi + \lambda^{-2}(-2\lambda^{-2}d_*' \\ &\qquad + \lambda^{-2}c_*)\,\chi' - \lambda^{-4}d_*\chi'', \\ v_2 &= \lambda^{-2}\chi'' + z\chi + \lambda^{-2}(-\lambda^{-2}c_* + \lambda^{-2}d_*')\,\chi + \lambda^{-4}d_*\chi', \\ v_3 &= -\chi' - \lambda^{-2}d_*\chi, \\ v_4 &= \chi. \end{aligned}\right\} \tag{2.68}$$

Hence, using the fundamental matrix solution

$$\boldsymbol{X} = \boldsymbol{H}_{(2m)}\boldsymbol{U} + \lambda^{-2m-2}\boldsymbol{E}(z,\lambda) \times \boldsymbol{R}(z,\lambda)$$

of $\boldsymbol{\chi}' = \boldsymbol{N}\boldsymbol{\chi}$ as given in theorem 2.4, we can now construct a fundamental matrix solution $\boldsymbol{V}$ of $\boldsymbol{v}' = -(\boldsymbol{N}^*)^T\,\boldsymbol{v}$. We obtain

$$\boldsymbol{V} = \boldsymbol{K}\boldsymbol{U} + \lambda^{-2m-2}\boldsymbol{E}(z,\lambda) \times \boldsymbol{S}(z,\lambda), \tag{2.69}$$

where the matrix $\boldsymbol{K} \in \Lambda$, with

$$\boldsymbol{K}_0 = \begin{bmatrix} -1 & -z & 0 & -1 \\ z & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{2.70}$$

since $(\boldsymbol{H}_{(2m)})_0 = I$ by (2.48), and where

$$\boldsymbol{S} = \begin{bmatrix} 1 & \lambda^{\frac{2}{3}}\ln\lambda & \lambda^{\frac{2}{3}}\ln\lambda & \lambda^{\frac{2}{3}}\ln\lambda \\ \lambda^{-\frac{2}{3}} & \ln\lambda & \ln\lambda & \ln\lambda \\ 1 & \lambda^{\frac{4}{3}}\ln\lambda & \lambda^{\frac{4}{3}}\ln\lambda & \lambda^{\frac{4}{3}}\ln\lambda \\ 1 & \lambda^{\frac{2}{3}}\ln\lambda & \lambda^{\frac{2}{3}}\ln\lambda & \lambda^{\frac{2}{3}}\ln\lambda \end{bmatrix} \quad \text{for} \quad z \in \Omega, \tag{2.71a}$$

$$\boldsymbol{S} = \begin{bmatrix} 1 & \lambda^{\frac{3}{2}}\ln\lambda\, z^{\frac{5}{4}}\mathrm{e}^{-\xi} & \lambda^{\frac{3}{2}}\ln\lambda\, z^{\frac{5}{4}}\mathrm{e}^{\xi} & \ln\lambda\, z^{-1} \\ z & \lambda^{\frac{1}{2}}\ln\lambda\, z^{\frac{3}{4}}\mathrm{e}^{-\xi} & \lambda^{\frac{1}{2}}\ln\lambda\, z^{\frac{3}{4}}\mathrm{e}^{\xi} & \ln\lambda \\ 1 & \lambda^{\frac{3}{2}}\ln\lambda\, z^{\frac{1}{4}}\mathrm{e}^{-\xi} & \lambda^{\frac{3}{2}}\ln\lambda\, z^{\frac{1}{4}}\mathrm{e}^{\xi} & \ln\lambda\, z^{-2} \\ 1 & \lambda^{\frac{1}{2}}\ln\lambda\, z^{-\frac{1}{4}}\mathrm{e}^{-\xi} & \lambda^{\frac{1}{2}}\ln\lambda\, z^{-\frac{1}{4}}\mathrm{e}^{\xi} & \ln\lambda\, z^{-1} \end{bmatrix} \quad \text{for} \quad z \in D_3' \backslash \Omega, \tag{2.71b}$$

by virtue of the definition (2.62) of the matrix $\boldsymbol{R}$.

(*b*) Since $\boldsymbol{V}$ is a fundamental matrix solution of $\boldsymbol{v}' = -(\boldsymbol{N}^*)^T\,\boldsymbol{v}$, it can be verified that the matrix

$$\boldsymbol{\Phi} = (\boldsymbol{V}^T)^{-1}$$

is a fundamental matrix solution of $\boldsymbol{\phi}' = \boldsymbol{N}^*\boldsymbol{\phi}$. Thus, from (2.69), we have

$$\begin{aligned} \boldsymbol{\Phi} &= (\boldsymbol{U}^T\boldsymbol{K}^T + \lambda^{-2m-2}\boldsymbol{E}(z,\lambda) \times \boldsymbol{S}^T)^{-1} \\ &= \{I + \lambda^{-2m-2}(\boldsymbol{K}^T)^{-1}(\boldsymbol{U}^T)^{-1}(\boldsymbol{E}(z,\lambda) \times \boldsymbol{S}^T)\}^{-1}(\boldsymbol{K}^T)^{-1}(\boldsymbol{U}^T)^{-1}. \end{aligned} \tag{2.72}$$

In order to calculate the matrix $(\boldsymbol{K}^T)^{-1}(\boldsymbol{U}^T)^{-1}$ explicitly, we first use the fact that $\boldsymbol{K} \in \Lambda$, with $\boldsymbol{K}_0$ given by (2.70), to deduce that $(\boldsymbol{K}^T)^{-1} \in \Lambda$, with

$$((\boldsymbol{K}^T)^{-1})_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & z \\ 1 & 0 & -z & -1 \end{bmatrix}. \tag{2.73}$$

Next, we write

$$(\boldsymbol{U}^T)^{-1} = (u_{ij}), \tag{2.74}$$

and note from the definition (2.59) of the matrix $\boldsymbol{U}$ that

$$\det \boldsymbol{U} = \lambda^{-4} W(B_0, A_1, A_2, B_3) = 4\pi^2 \mathrm{e}^{-2\pi\mathrm{i}\alpha} \beta_0^{\alpha-1} \lambda^{2\alpha-2}, \tag{2.75}$$

where we have inserted the value of the Wronskian as given in appendix A (A 8). We now express the elements u_{ij} in (2.74) in terms of certain third order Wronskians; in particular, we find that

$$\left.\begin{aligned} u_{41} &= -\lambda^{-2} (\det \boldsymbol{U})^{-1} W(A_1, A_2, B_3), \\ u_{42} &= \lambda^{-2} (\det \boldsymbol{U})^{-1} W(B_0, A_2, B_3), \\ u_{43} &= -\lambda^{-2} (\det \boldsymbol{U})^{-1} W(B_0, A_1, B_3), \\ u_{44} &= \lambda^{-2} (\det \boldsymbol{U})^{-1} W(B_0, A_1, A_2). \end{aligned}\right\} \tag{2.76}$$

However, third order Wronskians of the type appearing in (2.76) have been studied by Rabenstein (1959). From § 4 of that paper we have that, with

$$A_j \equiv A_j(z; \alpha, \beta_0, \lambda); \quad A_j^* \equiv A_j(z; 2-\alpha, \beta_0, \lambda),$$

and with similar meanings for B_j and B_j^*,

$$\left.\begin{aligned} W(A_1, A_2, B_3) &= -2\pi\mathrm{i}\, \mathrm{e}^{-3\pi\mathrm{i}\alpha} \beta_0^{\alpha-1} \lambda^{2\alpha} B_3^*, \\ W(B_0, A_2, B_3) &= 2\pi\mathrm{i}\, \mathrm{e}^{-\pi\mathrm{i}\alpha} \beta_0^{\alpha-1} \lambda^{2\alpha} A_2^*, \\ W(B_0, A_1, B_3) &= 2\pi\mathrm{i}\, \mathrm{e}^{-3\pi\mathrm{i}\alpha} \beta_0^{\alpha-1} \lambda^{2\alpha} A_1^*, \\ W(B_0, A_1, A_2) &= 2\pi\mathrm{i}\, \mathrm{e}^{-\pi\mathrm{i}\alpha} \beta_0^{\alpha-1} \lambda^{2\alpha} \{(1-\mathrm{e}^{-2\pi\mathrm{i}\alpha}) B_3^* - B_0^*\}. \end{aligned}\right\} \tag{2.77}$$

Thus, by (2.75) and (2.76),

$$\left.\begin{aligned} u_{41} = (\mathrm{i}/2\pi)\, \mathrm{e}^{-\pi\mathrm{i}\alpha} B_3^*, \quad u_{42} = (\mathrm{i}/2\pi)\, \mathrm{e}^{\pi\mathrm{i}\alpha} A_2^*, \\ u_{43} = -(\mathrm{i}/2\pi)\, \mathrm{e}^{-\pi\mathrm{i}\alpha} A_1^*, \quad u_{44} = -(\mathrm{i}/2\pi)\, \mathrm{e}^{\pi\mathrm{i}\alpha} \{B_0^* - (1-\mathrm{e}^{-2\pi\mathrm{i}\alpha})\, B_3^*\}. \end{aligned}\right\} \tag{2.78}$$

Next, we use repeatedly the fact that B_0, A_1, A_2, B_3 are solutions of $\mathscr{Q}u = 0$, together with (2.75) and (2.77), to obtain

$$\begin{aligned} u_{31} &= \lambda^{-2} (\det \boldsymbol{U})^{-1} \begin{vmatrix} A_1 & A_2 & B_3 \\ A_1' & A_2' & B_3' \\ A_1''' & A_2''' & B_3''' \end{vmatrix} \\ &= \lambda^{-2} (\det \boldsymbol{U})^{-1} \{W(A_1, A_2, B_3)\}' \\ &= -(\mathrm{i}/2\pi)\, \mathrm{e}^{-\pi\mathrm{i}\alpha} B_3^{*\prime}, \end{aligned} \tag{2.79a}$$

$$\begin{aligned} u_{21} &= -\lambda^{-4} (\det \boldsymbol{U})^{-1} \begin{vmatrix} A_1 & A_2 & B_3 \\ A_1'' & A_2'' & B_3'' \\ A_1''' & A_2''' & B_3''' \end{vmatrix} \\ &= -\lambda^{-4} (\det \boldsymbol{U})^{-1} [\{W(A_1, A_2, B_3)\}'' + \lambda^2 z W(A_1, A_2, B_3)] \\ &= (\mathrm{i}/2\pi)\, \mathrm{e}^{-\pi\mathrm{i}\alpha} (\lambda^{-2} B_3^{*\prime\prime} + z B_3^*), \end{aligned} \tag{2.79b}$$

$$\begin{aligned} u_{11} &= \lambda^{-4} (\det \boldsymbol{U})^{-1} \begin{vmatrix} A_1' & A_2' & B_3' \\ A_1'' & A_2'' & B_3'' \\ A_1''' & A_2''' & B_3''' \end{vmatrix} \\ &= \lambda^{-4} (\det \boldsymbol{U})^{-1} [\{W(A_1, A_2, B_3)\}''' + \lambda^2 z \{W(A_1, A_2, B_3)\}' \\ &\quad + \lambda^2 (1-\alpha) W(A_1, A_2, B_3)] \\ &= -(\mathrm{i}/2\pi)\, \mathrm{e}^{-\pi\mathrm{i}\alpha} (\lambda^{-2} B_3^{*\prime\prime\prime} + z B_3^{*\prime} + (1-\alpha)\, B_3^*). \end{aligned} \tag{2.79c}$$

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The remaining elements u_{ij} of the matrix $(\boldsymbol{U}^T)^{-1}$ are computed in a similar manner. Then, setting $\alpha_0 = 2$ in (2.78) and (2.79), we deduce from the form (2.73) of the matrix $((\boldsymbol{K}^T)^{-1})_0$ that

$$(\boldsymbol{K}^T)^{-1}(\boldsymbol{U}^T)^{-1} = -(\mathrm{i}/2\pi)\,\boldsymbol{P}(z,\lambda)\begin{bmatrix} B_3^* & A_2^* & -A_1^* & -B_0^*+c(\lambda)\,B_3^* \\ B_3^{*\prime} & A_2^{*\prime} & -A_1^{*\prime} & -B_0^{*\prime}+c(\lambda)\,B_3^{*\prime} \\ \lambda^{-2}B_3^{*\prime\prime} & \lambda^{-2}A_2^{*\prime\prime} & -\lambda^{-2}A_1^{*\prime\prime} & \lambda^{-2}\{-B_0^{*\prime\prime}+c(\lambda)\,B_3^{*\prime\prime}\} \\ \lambda^{-2}B_3^{*\prime\prime\prime} & \lambda^{-2}A_2^{*\prime\prime\prime} & -\lambda^{-2}A_1^{*\prime\prime\prime} & \lambda^{-2}\{-B_0^{*\prime\prime\prime}+c(\lambda)\,B_3^{*\prime\prime\prime}\} \end{bmatrix}, \tag{2.80a}$$

where the matrix $\boldsymbol{P} \in \mathit{\Lambda}$, with

$$\boldsymbol{P}_0 = \boldsymbol{I}, \tag{2.80b}$$

and

$$c(\lambda) = O(\lambda^{-2}) \quad \text{for} \quad |\lambda| \to \infty. \tag{2.80c}$$

Next, using (A 2) to (A 7) in appendix A to bound the functions B_0^*, A_1^*, A_2^*, B_3^* and their derivatives as they appear in the right member of (2.80*a*), we find that

$$(\boldsymbol{K}^T)^{-1}(\boldsymbol{U}^T)^{-1} = \boldsymbol{E}(z,\lambda)\times\begin{bmatrix} 1 & \lambda^{-\frac{2}{3}} & \lambda^{-\frac{2}{3}} & \lambda^{-\frac{2}{3}} \\ \ln\lambda & 1 & 1 & 1 \\ \lambda^{-\frac{4}{3}} & \lambda^{-\frac{4}{3}} & \lambda^{-\frac{4}{3}} & \lambda^{-2} \\ \lambda^{-\frac{2}{3}} & \lambda^{-\frac{2}{3}} & \lambda^{-\frac{2}{3}} & \lambda^{-2} \end{bmatrix} \quad \text{for } z\in\Omega, \tag{2.81a}$$

$$(\boldsymbol{K}^T)^{-1}(\boldsymbol{U}^T)^{-1} = \boldsymbol{E}(z,\lambda)\times\begin{bmatrix} 1 & \lambda^{-\frac{3}{2}}z^{-\frac{5}{4}}\mathrm{e}^{\xi} & \lambda^{-\frac{3}{2}}z^{-\frac{5}{4}}\mathrm{e}^{-\xi} & z \\ \ln z & \lambda^{-\frac{1}{2}}z^{-\frac{3}{4}}\mathrm{e}^{\xi} & \lambda^{-\frac{1}{2}}z^{-\frac{3}{4}}\mathrm{e}^{-\xi} & 1 \\ \lambda^{-2}z^{-1} & \lambda^{-\frac{3}{2}}z^{-\frac{1}{4}}\mathrm{e}^{\xi} & \lambda^{-\frac{3}{2}}z^{-\frac{1}{4}}\mathrm{e}^{-\xi} & \lambda^{-2} \\ \lambda^{-2}z^{-2} & \lambda^{-\frac{1}{2}}z^{\frac{1}{4}}\mathrm{e}^{\xi} & \lambda^{-\frac{1}{2}}z^{\frac{1}{4}}\mathrm{e}^{-\xi} & \lambda^{-2} \end{bmatrix} \quad \text{for} \quad z\in D_3'\backslash\Omega. \tag{2.81b}$$

Now, by virtue of the definition (2.71) of the matrix $\boldsymbol{S}$, and the estimates (2.81), an explicit calculation yields

$$(\boldsymbol{K}^T)^{-1}(\boldsymbol{U}^T)^{-1}\boldsymbol{S}^T = \ln\lambda\,\boldsymbol{E}(z,\lambda)\times\begin{bmatrix} 1 & \lambda^{-\frac{2}{3}} & \lambda^{\frac{2}{3}} & 1 \\ \lambda^{\frac{2}{3}} & 1 & \lambda^{\frac{4}{3}} & \lambda^{\frac{2}{3}} \\ \lambda^{-\frac{2}{3}} & \lambda^{-\frac{4}{3}} & 1 & \lambda^{-\frac{2}{3}} \\ 1 & \lambda^{-\frac{2}{3}} & \lambda^{\frac{2}{3}} & 1 \end{bmatrix} \quad \text{for} \quad z\in\Omega, \tag{2.82a}$$

$$(\boldsymbol{K}^T)^{-1}(\boldsymbol{U}^T)^{-1}\boldsymbol{S}^T = \ln\lambda\,\boldsymbol{E}(z,\lambda)\times\begin{bmatrix} 1 & z & z^{-1} & 1 \\ \lambda z^{\frac{1}{2}} & 1 & \lambda z^{-\frac{1}{2}} & z^{-1} \\ z & \lambda^{-1}z^{\frac{1}{2}} & 1 & \lambda^{-1}z^{-\frac{1}{2}} \\ \lambda z^{\frac{3}{2}} & z & \lambda z^{\frac{1}{2}} & 1 \end{bmatrix} \quad \text{for} \quad z\in D_3'\backslash\Omega. \tag{2.82b}$$

(*c*) The fundamental matrix solution $\boldsymbol{\Phi}$ of $\boldsymbol{\phi}' = \boldsymbol{N}^*\boldsymbol{\phi}$ is now given by (2.72):

$$\boldsymbol{\Phi} = (\boldsymbol{K}^T)^{-1}(\boldsymbol{U}^T)^{-1} + \lambda^{-2m-2}\ln\lambda\,\boldsymbol{E}(z,\lambda)\times\boldsymbol{T}(z,\lambda), \tag{2.83a}$$

where, using the estimates (2.81) and (2.82), we can show that

$$\boldsymbol{T} = \begin{bmatrix} 1 & \lambda^{-\frac{2}{3}} & \lambda^{-\frac{2}{3}} & \lambda^{-\frac{2}{3}} \\ \lambda^{\frac{2}{3}} & 1 & 1 & 1 \\ \lambda^{-\frac{2}{3}} & \lambda^{-\frac{4}{3}} & \lambda^{-\frac{4}{3}} & \lambda^{-\frac{4}{3}} \\ 1 & \lambda^{-\frac{2}{3}} & \lambda^{-\frac{2}{3}} & \lambda^{-\frac{2}{3}} \end{bmatrix} \quad \text{for} \quad z\in\Omega, \tag{2.83b}$$

$$\boldsymbol{T} = \begin{bmatrix} 1 & \lambda^{-\frac{1}{2}}z^{\frac{1}{4}}\mathrm{e}^{\xi} & \lambda^{-\frac{1}{2}}z^{\frac{1}{4}}\mathrm{e}^{-\xi} & z \\ \lambda z^{\frac{1}{2}} & \lambda^{-\frac{1}{2}}z^{-\frac{3}{4}}\mathrm{e}^{\xi} & \lambda^{-\frac{1}{2}}z^{-\frac{3}{4}}\mathrm{e}^{-\xi} & \lambda z^{\frac{3}{2}} \\ z & \lambda^{-\frac{3}{2}}z^{-\frac{1}{4}}\mathrm{e}^{\xi} & \lambda^{-\frac{3}{2}}z^{-\frac{1}{4}}\mathrm{e}^{-\xi} & z^2 \\ \lambda z^{\frac{3}{2}} & \lambda^{-\frac{1}{2}}z^{\frac{1}{4}}\mathrm{e}^{\xi} & \lambda^{-\frac{1}{2}}z^{\frac{1}{4}}\mathrm{e}^{-\xi} & \lambda z^{\frac{5}{2}} \end{bmatrix} \quad \text{for} \quad z\in D_3'\backslash\Omega. \tag{2.83c}$$

The result (2.64) of the theorem now follows from (2.83) if we set $m = 0$, write

$$\boldsymbol{\Phi} = \begin{bmatrix} \phi_3 & \phi_2 & -\phi_1 & -\phi_0 + c(\lambda)\,\phi_3 \\ \phi_3' & \phi_2' & -\phi_1' & -\phi_0' + c(\lambda)\,\phi_3' \\ \lambda^{-2}\phi_3'' & \lambda^{-2}\phi_2'' & -\lambda^{-2}\phi_1'' & \lambda^{-2}(-\phi_0'' + c(\lambda)\,\phi_3'') \\ \lambda^{-2}\phi_3''' & \lambda^{-2}\phi_2''' & -\lambda^{-2}\phi_1''' & \lambda^{-2}(-\phi_0''' + c(\lambda)\,\phi_3''') \end{bmatrix}, \tag{2.84}$$

with $c(\lambda)$ as in (2.80), and then insert the relevant asymptotic properties of the functions

$$B_0^*|_{m=0}(\equiv B_0(z;0,\beta_0,\lambda)), \quad A_1^*|_{m=0}, A_2^*|_{m=0} \quad \text{and} \quad B_3^*|_{m=0}$$

(as given in appendix A) into the expression (2.80) of the matrix $(\boldsymbol{K}^T)^{-1}\,(\boldsymbol{U}^T)^{-1}$. Similarly, if we set $m = 1$ in (2.80), we obtain the estimates (2.65) of the theorem.

(*d*) Consider the asymptotic expansions (A 2) and (A 6) of the functions

$$B_0^*\,(\equiv B_0(z;2-\alpha,\beta_0,\lambda)) \quad \text{and} \quad B_3^*.$$

The result (2.66) now follows from (2.80), (2.83) and (2.84) if we expand each term in these expansions of B_0^* and B_3^* as a Taylor series about $\alpha = \alpha_0$.

(*e*) Finally, by virtue of the relation (2.68) between the fundamental matrices $\boldsymbol{X}$ and $\boldsymbol{V}$ of $\boldsymbol{\chi}' = \boldsymbol{N}\boldsymbol{\chi}$ and $\boldsymbol{v}' = -(\boldsymbol{N}^*)^T\boldsymbol{v}$, we can deduce that $\det \boldsymbol{X} = \det \boldsymbol{V}$. Consequently, since

$$\boldsymbol{\Phi} = (\boldsymbol{V}^T)^{-1},$$

we have that $\quad \det\boldsymbol{\Phi} = (\det \boldsymbol{V})^{-1} = (\det \boldsymbol{X})^{-1} \sim (4\pi^2)^{-1}\beta_0^{-1}\lambda^{-2} \quad \text{for} \quad |\lambda| \to \infty$

by (2.63). The result (2.67) now follows from (2.84).

3. Bounds for certain integrals

(*a*) *Preliminaries*

In the proofs of the theorems of §§ 4 and 5, estimates of certain integrals will be crucial. To this end, we define the sets S_j ($j = 1, 2, 3$; see figure 2), to be like the sets S_j' except that the radii R_j' are replaced by the small radii R_j, where

$$z_b < R_1 < R_1', \quad -z_a < R_2 < R_2' \quad \text{and} \quad R_3 = \tfrac{1}{2}R_3'.$$

We also write
$$S = S_1 \cup S_2 \cup S_3, \quad \bar{D}_3 = \bar{\Omega} \cup \bar{S}, \quad D_2 = \operatorname{int} \bar{D}_3. \tag{3.1}$$

Furthermore, let z_* be any point such that

$$z_* \in S_1, \quad |z_*| > R_3, \tag{3.2}$$

and define the points z_r and z_l by

$$z_r \in \partial S_1', \quad |z_r| = R_1', \quad \arg z_r = -\delta_1, \tag{3.3}$$

$$z_l \in \partial S_2', \quad |z_l| = R_2', \quad \arg z_l = -\pi + \delta_1, \tag{3.4}$$

where $\partial S_1'$, $\partial S_2'$ denote the boundaries of S_1', S_2'.

Now introduce
$$\xi(z,\lambda) = \tfrac{2}{3}\mathrm{i}\lambda z^{\frac{3}{2}} \tag{3.5}$$

(cf. (2.60)), and define a depleted image of Ω by

$$W = \{\xi \mid -\tfrac{5}{4}\pi - \delta_2 < \arg \xi < \tfrac{1}{4}\pi + \delta_2, \quad |\xi| < N\}, \tag{3.6}$$

where $\delta_2 = \delta + (\frac{3}{2})\,\delta_1 < \frac{1}{4}\pi$ by (2.57), and where $N = (\frac{2}{3})\,K^{\frac{3}{2}}$, with K the number occurring in the definition (2.54) of Ω. Suppose that the transformation (3.5) maps the sets S_1, S_2 and S_3 in the z-plane on to the sets (figure 3)

$$\left.\begin{aligned} T_1 &= \{\xi \,|\, |\arg\xi - \tfrac{1}{4}\pi| < \delta_2, \quad N \leqslant |\xi| < K_1|\lambda|\},\\ T_2 &= \{\xi \,|\, |\arg\xi + \tfrac{5}{4}\pi| < \delta_2, \quad N \leqslant |\xi| < K_2|\lambda|\},\\ T_3 &= \{\xi \,|\, -\tfrac{5}{4}\pi + \delta_2 < \arg\xi < \tfrac{1}{4}\pi - \delta_2, \quad N \leqslant |\xi| < K_3|\lambda|\}\end{aligned}\right\} \tag{3.7}$$

(respectively) in the ξ-plane, where, in terms of the numbers defining the S_j, we have

$$K_j = \tfrac{2}{3}(R_j)^{\frac{3}{2}} \quad (j = 1, 2, 3).$$

The S'_j are mapped similarly on to sets T'_j, with K'_j replacing K_j. We define the sets

$$T' = T'_1 \cup T'_2 \cup T'_3, \quad \overline{\mathscr{D}}' = \overline{W} \cup \overline{T}', \quad \mathscr{D}' = \operatorname{int}\overline{\mathscr{D}}', \tag{3.8}$$

$$T = T_1 \cup T_2 \cup T_3, \quad \overline{\mathscr{D}} = \overline{W} \cup \overline{T}, \quad \mathscr{D} = \operatorname{int}\overline{\mathscr{D}}, \tag{3.9}$$

and the points $\xi_*, \xi_r, \xi_l \in \overline{\mathscr{D}}'$ by $\xi_* \equiv \xi(z_*, \lambda)$, $\xi_r \equiv \xi(z_r, \lambda)$ and $\xi_l \equiv \xi(z_l, \lambda)$, with $z_*, z_r, z_l \in \overline{D}'_3$ as in (3.2), (3.3) and (3.4).

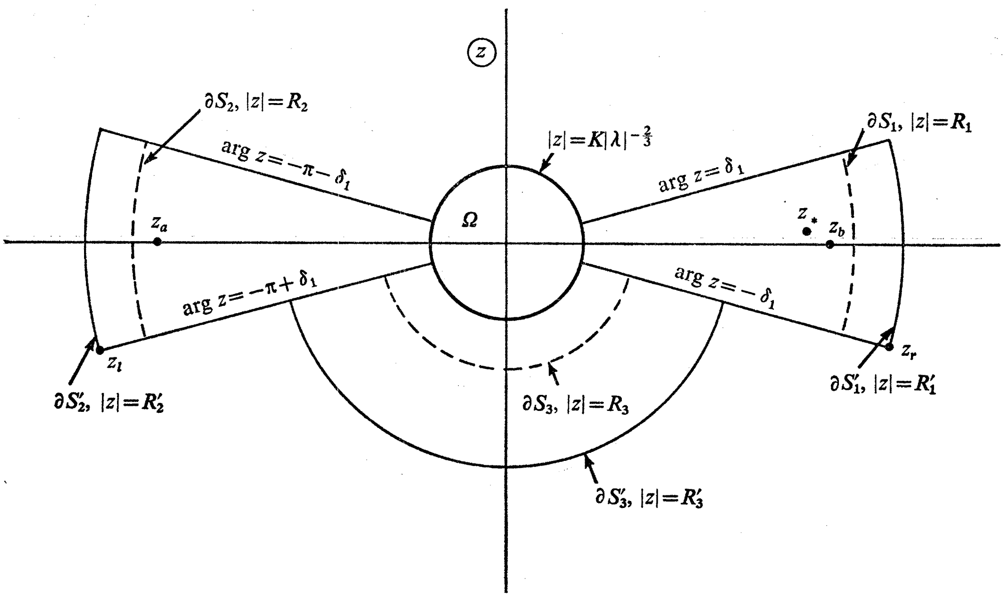

FIGURE 2. The domain $D_3 (\subset D'_3)$ in the z-plane as defined by (3.1). The points z_*, z_r and z_l are defined by (3.2)–(3.4).

Remark. The basic reason for the complicated nature of the domains constructed above, as opposed to the much simpler *disks* which were used in the analogous §6 of Lin & Rabenstein's (1969) paper, is that here the domains of interest in the z-plane *must* contain the real interval $[z_a, z_b]$, whereas in their work no such condition was necessary.

Next, we introduce the *truncation operator* tru defined by

$$\operatorname{tru}|\xi|^k = \begin{cases} |\xi|^k & \text{for} \quad |\xi| \geqslant N, \\ N^k & \text{for} \quad |\xi| \leqslant N, \end{cases} \tag{3.10}$$

where k denotes any real number.

Let M be a generic symbol for fixed positive numbers (independent of z and λ), and let η be an integration variable in the ξ-plane. *In lemmas* 3.1 *to* 3.5 *it is to be understood that*

(*a*) *ξ is any point of $\mathscr{D}$,*

(*b*) *the result is claimed only for the paths of integration chosen in the proof,*

(*c*) $\int_{\xi_*}^{\xi} |f(\eta)|$ *stands for* $\int_{s_*}^{s} |f(\eta(t))||\eta'(t)|\,\mathrm{d}t$, *where* $\eta = \eta(t)$, $s_* \leqslant t \leqslant s$ *is a parametric equation of the path in question, directed from* $\xi_* = \eta(s_*)$ *to* $\xi = \eta(s)$.

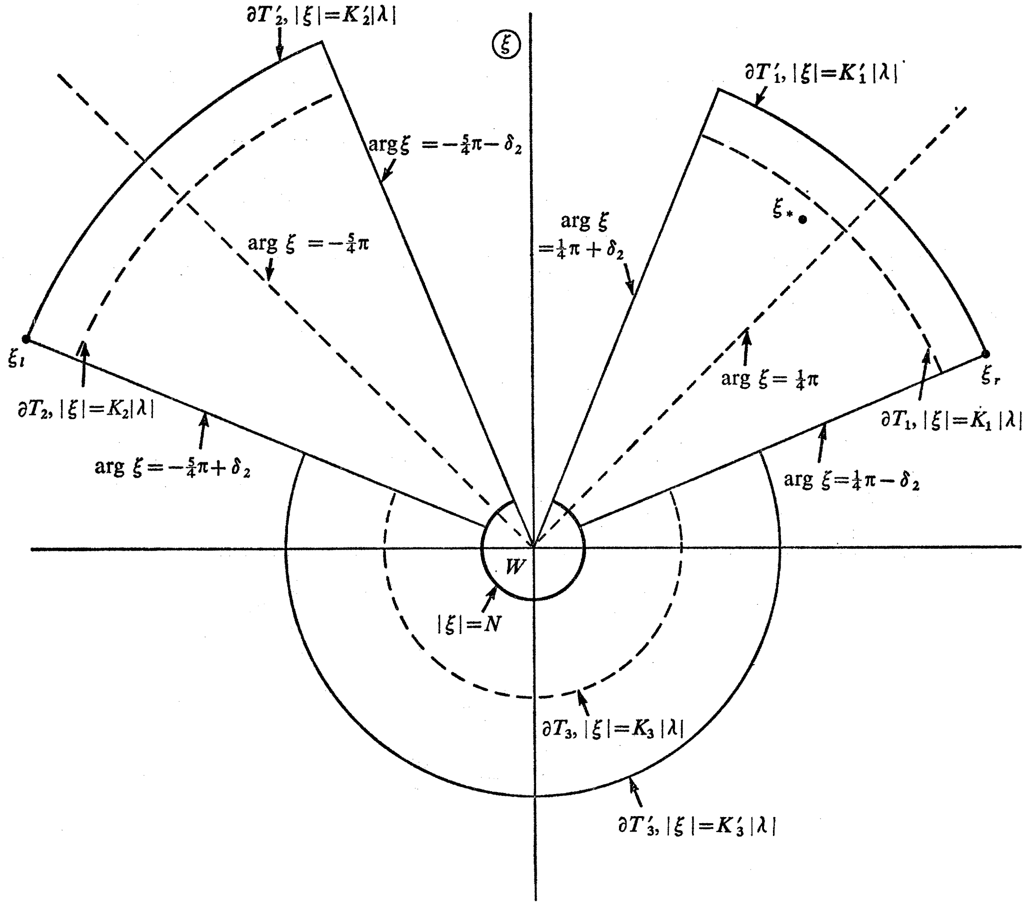

FIGURE 3. The sets $\mathscr{D}$ and $\mathscr{D}'$ in the ξ-plane as defined by (3.8) and (3.9). The points ξ_*, ξ_r and ξ_l are also shown.

(*b*) *Integral estimates for the proof of theorem* 4.1

The following two lemmas are merely stated, since their proofs are similar to, but simpler than, those of lemma 3.3 and 3.5.

LEMMA 3.1. *For any fixed real number $k \leqslant -1$,*

$$\int_{\xi_*}^{\xi} |(\mathrm{tru}\,|\eta|^{k+\frac{1}{3}})\,\eta^{-\frac{1}{3}}| \leqslant \begin{cases} M\,\mathrm{tru}\,|\xi|^{k+1} & \textit{for} \quad k < -1, \\ M \ln|\lambda| & \textit{for} \quad k = -1. \end{cases}$$

LEMMA 3.2. *For any fixed real number k*,

$$\int_{\xi}^{\xi_r} |(\operatorname{tru}|\eta|^{k+\frac{1}{3}})\,\eta^{-\frac{1}{3}}e^{-\eta}| \leqslant M \operatorname{tru}|\xi|^k|e^{-\xi}|,$$

and

$$\int_{\xi_l}^{\xi} |(\operatorname{tru}|\eta|^{k+\frac{1}{3}})\,\eta^{-\frac{1}{3}}e^{\eta}| \leqslant M \operatorname{tru}|\xi|^k|e^{\xi}|.$$

(*c*) *Integral estimates for the proof of theorem* 5.1

First, suppose that $F(\xi,\lambda)$ is a function that is continuous for $\xi\in\mathscr{D}'$, and such that the estimates

$$|F(\xi,\lambda)| \leqslant \begin{cases} M\operatorname{tru}|\xi|^{\frac{1}{6}} & \textit{for} \quad |\xi| \leqslant |\lambda|^{\frac{1}{4}}, \\ M|\lambda|^{\frac{2}{3}}|\xi|^{-\frac{5}{2}} & \textit{for} \quad |\xi| \geqslant |\lambda|^{\frac{1}{4}}, \end{cases} \tag{3.11}$$

hold for $\xi\in\overline{\mathscr{D}}'$.

LEMMA 3.3. *For any fixed real number k such that* $-\frac{7}{6}<k<\frac{3}{2}$,

$$\int_{\xi_*}^{\xi} |(\operatorname{tru}|\eta|^{k+\frac{1}{3}})\,\eta^{-\frac{1}{3}}F(\eta,\lambda)| \leqslant \begin{cases} M|\lambda|^{\frac{1}{4}(k+\frac{7}{6})} & \textit{for} \quad |\xi| \leqslant |\lambda|^{\frac{1}{4}}, \\ M|\lambda|^{\frac{2}{3}}|\xi|^{k-\frac{3}{2}} & \textit{for} \quad |\xi| \geqslant |\lambda|^{\frac{1}{4}}. \end{cases}$$

Proof. We introduce the notation

$$|\lambda| = l,\ |\eta| = \rho,\ |\xi| = r \quad \text{and} \quad G(\eta,\lambda) = |\,(\operatorname{tru}|\eta|^{k+\frac{1}{3}})\,\eta^{-\frac{1}{3}}F(\eta,\lambda)|.$$

Then, by (3.11),

$$G(\eta,\lambda) \leqslant \begin{cases} M\rho^{-\frac{1}{3}} & \text{for} \quad \rho\leqslant N, \\ M\rho^{k+\frac{1}{6}} & \text{for} \quad N\leqslant\rho\leqslant l^{\frac{1}{4}}, \\ Ml^{\frac{2}{3}}\rho^{k-\frac{5}{2}} & \text{for} \quad \rho\geqslant l^{\frac{1}{4}}. \end{cases} \tag{3.12}$$

Also, $\xi_*\in T_1$, with $|\xi_*| > K_3 l$. Now, using (3.12), we bound *integrals along radial lines* as follows.

$$\int_{|\xi_*|}^{r} G(\eta,\lambda)\,d\rho \leqslant Ml^{\frac{2}{3}}l^{k-\frac{3}{2}} \quad \text{for} \quad r\geqslant|\xi_*|; \tag{3.13}$$

$$\int_{r}^{|\xi_*|} G(\eta,\lambda)\,d\rho \leqslant \begin{cases} Ml^{\frac{2}{3}}r^{k-\frac{3}{2}} & \text{for} \quad l^{\frac{1}{4}}\leqslant r\leqslant|\xi_*|, & (3.14)\\ Ml^{\frac{2}{3}}l^{\frac{1}{4}(k-\frac{3}{2})}+M(l^{\frac{1}{4}(k+\frac{7}{6})}-r^{k+\frac{7}{6}}) \leqslant Ml^{\frac{1}{4}(k+\frac{7}{6})} & \text{for} \quad N\leqslant r\leqslant l^{\frac{1}{4}}, & (3.15)\\ Ml^{\frac{1}{4}(k+\frac{7}{6})}+M \leqslant Ml^{\frac{1}{4}(k+\frac{7}{6})} & \text{for} \quad r\leqslant N. & (3.16) \end{cases}$$

For *integrals along circles* we have

$$\int_{\rho=r} G(\eta,\lambda)\,|d\eta| \leqslant \begin{cases} Ml^{\frac{2}{3}}r^{k-\frac{3}{2}} & \text{for} \quad l^{\frac{1}{4}}\leqslant r\leqslant K_3 l, & (3.17)\\ Mr^{k+\frac{7}{6}} \leqslant Ml^{\frac{1}{4}(k+\frac{7}{6})} & \text{for} \quad r\leqslant l^{\frac{1}{4}}. & (3.18) \end{cases}$$

Now consider the various positions of ξ.

(*a*) Suppose that $|\xi|\geqslant|\xi_*|$. For the worst case, $\xi\in T_2$, take a path as shown in figure 4, and use (3.14), (3.17), (3.14) and (3.13) for the successive paths. Then

$$\int_{\xi_*}^{\xi} G(\eta,\lambda) \leqslant Ml^{\frac{2}{3}}l^{k-\frac{3}{2}} \leqslant Ml^{\frac{2}{3}}|\xi|^{k-\frac{3}{2}}$$

since $\xi=O(\lambda)$, with $|\xi|\geqslant|\xi_*|>K_3 l$.

(*b*) Suppose that $K_3 l\leqslant|\xi|\leqslant|\xi_*|$. We need only *part* of the path used in (*a*), and the same estimates apply.

(*c*) Suppose that $l^{\frac{1}{4}} \leqslant |\xi| \leqslant K_3 l$. Take a path as shown in figure 5, and use (3.14) and (3.17). We obtain

$$\int_{\xi_*}^{\xi} G(\eta, \lambda) \leqslant M l^{\frac{2}{3}} |\xi|^{k-\frac{3}{2}}.$$

(*d*) Finally, suppose that $|\xi| \leqslant l^{\frac{1}{4}}$. Take a path as in (*c*), and use (3.15) or (3.16) and (3.18). It follows that

$$\int_{\xi_*}^{\xi} G(\eta, \lambda) \leqslant M l^{\frac{1}{4}(k+\frac{7}{6})}.$$

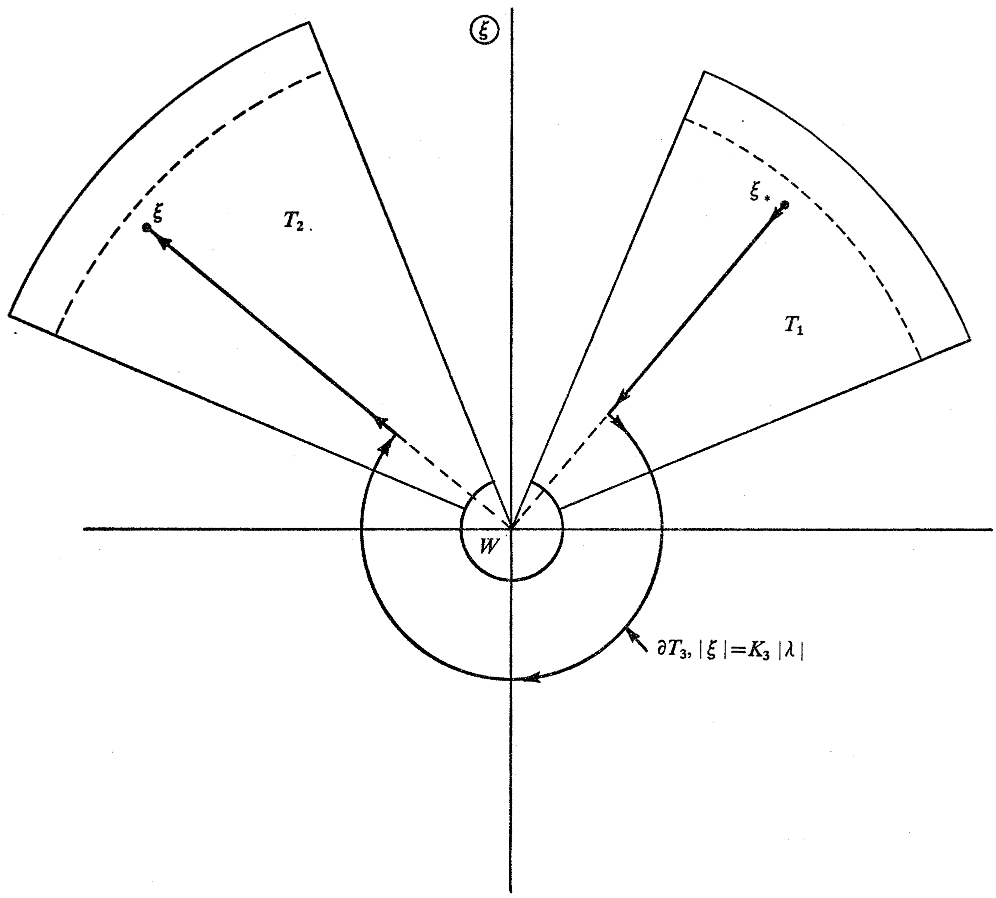

FIGURE 4. The path of integration used in lemma 3.3 for the case $|\xi| \geqslant |\xi_*|$, $\xi \in T_2$.

LEMMA 3.4. *With* $l = |\lambda|$ *and large, and* $\rho \geqslant 0$ *and real, define for any fixed real number* k (figure 6),

$$f(\rho) = \begin{cases} 1 & \textit{for} \quad \rho \leqslant 1, \\ \rho^{k+\frac{1}{6}} & \textit{for} \quad 1 \leqslant \rho \leqslant l^{\frac{1}{4}}, \\ l^{\frac{2}{3}} \rho^{k-\frac{5}{2}} & \textit{for} \quad \rho \geqslant l^{\frac{1}{4}}. \end{cases} \tag{3.19}$$

Then there exists a number $m > 0$ *depending only on* k *such that*

$$f(\rho) \leqslant f(r)\{1+|\rho-r|\}^m \quad \textit{for all} \quad \rho \geqslant 0 \quad \textit{and} \quad r \geqslant 0. \tag{3.20}$$

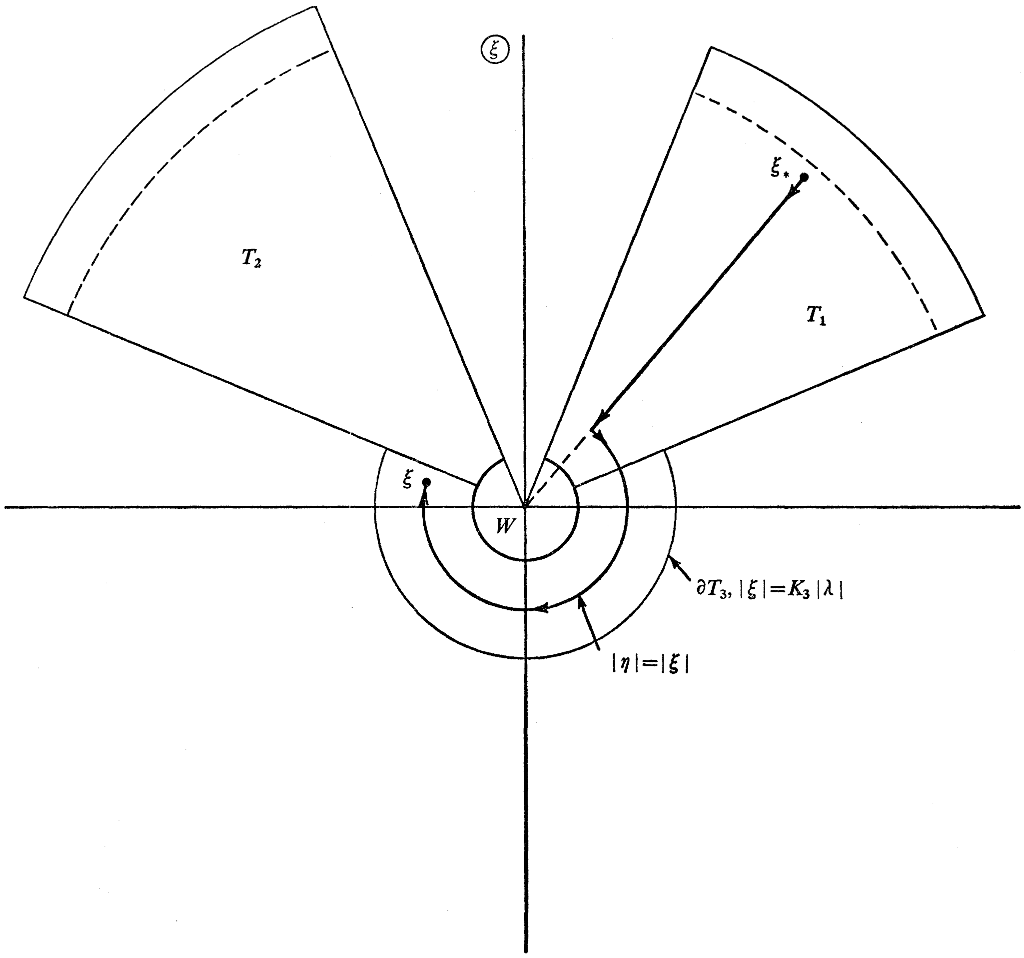

FIGURE 5. The path of integration used in lemma 3.3 for the case $|\lambda|^{\frac{1}{4}} \leqslant |\xi| \leqslant K_3|\lambda|$.

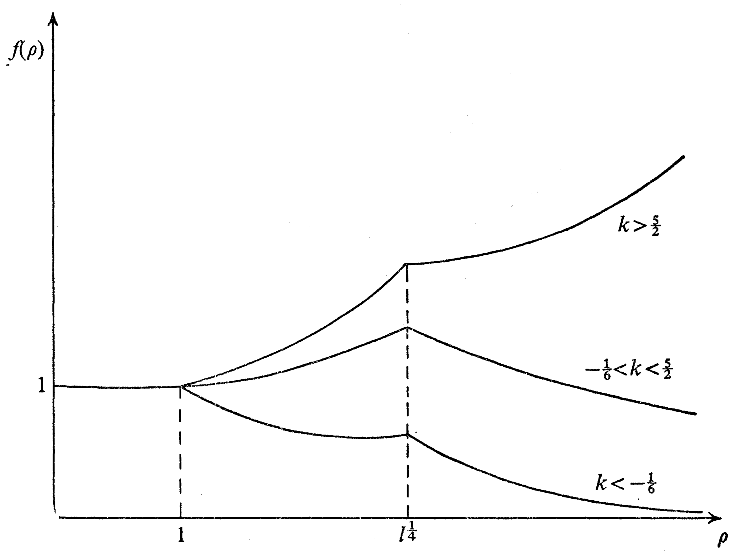

FIGURE 6. The function $f(\rho)$, as defined by (3.19), shown for different values of k.

Proof. First take $\rho \geqslant r$ and write $n = \max\{k+\frac{1}{6}, 0\}$.

(*a*) If $\rho \geqslant r \geqslant 1$, we have

$$f(\rho)/f(r) \leqslant (\rho/r)^n \quad (\text{with equality if } \rho \leqslant l^{\frac{1}{4}} \quad \text{and} \quad k > -\tfrac{1}{6})$$
$$\leqslant \{1+(\rho-r)/r\}^n \leqslant (1+\rho-r)^n \quad \text{since} \quad r \geqslant 1.$$

(*b*) If $\rho \geqslant 1 \geqslant r$, we have $f(r) = 1$, and by (*a*) with $r = 1$,

$$f(\rho) \leqslant (1+\rho-1)^n \leqslant (1+\rho-r)^n.$$

(*c*) If $1 \geqslant \rho \geqslant r$, $\qquad f(\rho)/f(r) = 1 \leqslant (1+\rho-r)^n.$

Now take $\rho \leqslant r$, write $-p = \min\{k-\frac{5}{2}, 0\}$, and argue as before to show that

$$f(\rho)/f(r) \leqslant (1+r-\rho)^p.$$

Finally, choose $m = \max\{n, p\}$.

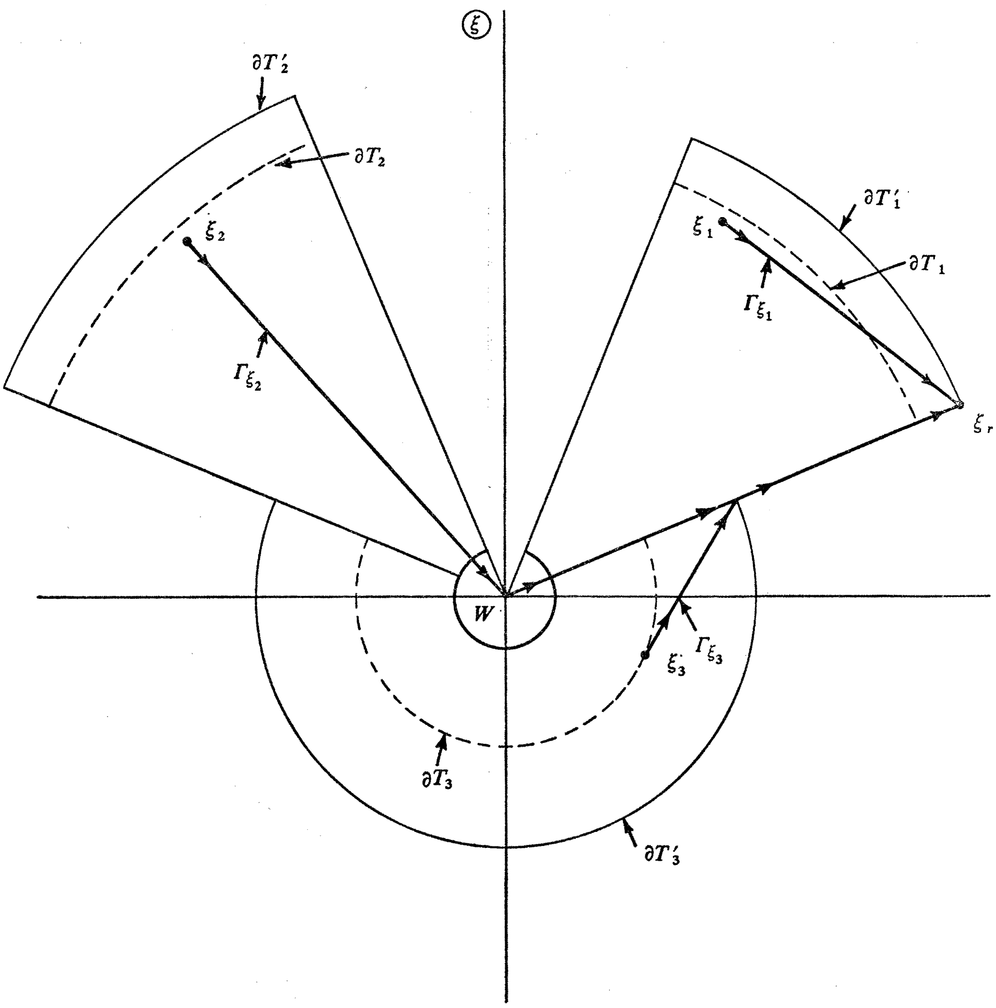

FIGURE 7. Examples of the integration path Γ_ξ used in lemma 3.5.

LEMMA 3.5. *With* $F(\xi, \lambda)$ *as in* (3.11), *and for any fixed real* k *and* p, $p > 0$,

$$\int_{\xi}^{\xi_r} |(\text{tru}\,|\eta|^{k+\frac{1}{3}})\, \eta^{-\frac{1}{3}} F(\eta, \lambda)\, \mathrm{e}^{-p\eta}| \leqslant \begin{cases} M\, \text{tru}\,|\xi|^{k+\frac{1}{6}}|\mathrm{e}^{-p\xi}| & \text{for} \quad |\xi| \leqslant |\lambda|^{\frac{1}{4}}, \\ M|\lambda|^{\frac{2}{3}}|\xi|^{k-\frac{5}{2}}|\mathrm{e}^{-p\xi}| & \text{for} \quad |\xi| \geqslant |\lambda|^{\frac{1}{4}}. \end{cases}$$

Proof. As before, we use the notation

$$|\lambda| = l, |\eta| = \rho, |\xi| = r \quad \text{and} \quad G(\eta,\lambda) = |(\text{tru}\,|\eta|^{k+\frac{1}{3}})\,\eta^{-\frac{1}{3}}F(\eta,\lambda)|,$$

where $G(\eta,\lambda)$ is bounded as in (3.12). Now, for each $\xi \in \mathscr{D}$, there exists a piecewise linear path, from ξ to ξ_r (figure 7),
$$\Gamma_\xi: \eta = \eta(s;\xi), \quad 0 \leqslant s \leqslant s_r(\xi), \quad \Gamma_\xi \subset \overline{\mathscr{D}}',$$
on which
$$(\partial/\partial s)\,\text{Re}\,(\eta) \geqslant h > 0; \tag{3.21}$$
here s is arc length and h is a fixed constant independent of ξ and l.

The first step is to dispose of $(\text{tru}\,|\eta|^{\frac{1}{3}})\,\eta^{-\frac{1}{3}}$ by proving that

$$\int_\xi^{\xi_r} G(\eta,\lambda)\,|\mathrm{e}^{-p\eta}| \leqslant M\int_\xi^{\xi_r} G_0(\rho,l)\,|\mathrm{e}^{-p\eta}|, \tag{3.22}$$

where the path of integration is Γ_ξ, and where

$$G_0(\rho,l) = \begin{cases} N^{k+\frac{1}{6}} & \text{for} \quad \rho \leqslant N, \\ \rho^{k+\frac{1}{6}} & \text{for} \quad N \leqslant \rho \leqslant l^{\frac{1}{4}}, \\ l^{\frac{2}{3}}\rho^{k-\frac{5}{2}} & \text{for} \quad \rho \geqslant l^{\frac{1}{4}}. \end{cases}$$

If ξ is such that $\Gamma_\xi \cap W = \varnothing$, (3.22) follows immediately. Suppose that $\xi \in W$, with $\text{Re}\,(\xi) \geqslant 0$, and that $\Gamma_\xi \cap W = \Gamma'_\xi$, so that Γ'_ξ is the straight line from ξ to a point $\xi_w \in \partial W$. Then, for (3.22) to be true, it suffices to prove that

$$\int_\xi^{\xi_w} G(\eta,\lambda)\,|\mathrm{e}^{-p\eta}| \leqslant M\int_\xi^{\xi_w} G_0(\rho,l)\,|\mathrm{e}^{-p\eta}|. \tag{3.23}$$

But
$$\int_\xi^{\xi_w} G_0(\rho,l)\,|\mathrm{e}^{-p\eta}| \geqslant N^{k+\frac{1}{6}} \min_{\eta\in\overline{W}} |\mathrm{e}^{-p\eta}|\,|\xi_w - \xi|,$$
and
$$\int_\xi^{\xi_w} G(\eta,\lambda)|\mathrm{e}^{-p\eta}| \leqslant \tfrac{3}{2}MN^{k+\frac{1}{2}} \max_{\eta\in\overline{W}} |\mathrm{e}^{-p\eta}|\,|\xi_w^{\frac{2}{3}} - \xi^{\frac{2}{3}}|.$$
Also,
$$|\xi_w^{\frac{2}{3}} - \xi^{\frac{2}{3}}| \leqslant M|\xi_w - \xi|$$
since $|\xi_w|\,(=N)$ is bounded away from zero, and (3.23) follows. A similar procedure yields (3.22) for all other points $\xi \in \mathscr{D}$ for which $\Gamma_\xi \cap W \neq \varnothing$.

Now, with $f(\rho) = f(\rho,l)$ as in (3.19), it can be verified that

$$\int_\xi^{\xi_r} G_0(\rho,l)\,|\mathrm{e}^{-p\eta}| \leqslant M\int_\xi^{\xi_r} f(\rho,l)|\mathrm{e}^{-p\eta}|.$$

Hence, if we can show that
$$\int_\xi^{\xi_r} f(\rho,l)\,|\mathrm{e}^{-p\eta}| \leqslant Mf(r,l)\,|\mathrm{e}^{-p\xi}|, \tag{3.24}$$
the result of the lemma follows.

Now on Γ_ξ we have $|\partial\rho/\partial s| \leqslant 1$; hence $|\rho - r| \leqslant s$ and, by (3.20),

$$f(\rho,l) \leqslant f(r,l)\,(1+s)^m. \tag{3.25}$$

Moreover, since $(\partial/\partial s)\,\text{Re}\,(\eta) \geqslant h > 0$ by (3.21), we have that

$$\text{Re}\,(\eta) \geqslant \text{Re}\,(\xi) + hs,$$

which implies that
$$|\mathrm{e}^{-p\eta}| \leqslant |\mathrm{e}^{-p\xi}|\,\mathrm{e}^{-phs}. \tag{3.26}$$

From the estimates (3.25) and (3.26) we now obtain (3.24), with

$$M = \int_0^\infty (1+s)^m\,\mathrm{e}^{-phs}\,\mathrm{d}s.$$

4. A uniformly bounded solution

Recall the definition (3.2) of the point $z_* \in D_3$:

$$z_* \in S_1, \quad |z_*| > R_3.$$

Then, with the operator $\mathscr{M}$ defined by

$$\mathscr{M}\psi \equiv \psi^{\mathrm{iv}} + \lambda^2\{p(z,\lambda)\,\psi'' + q(z,\lambda)\,\psi' + r(z,\lambda)\,\psi\}$$

as in (2.3), we prove

Theorem 4.1. *Suppose that there exists a function $v(.,\lambda) \in \mathscr{H}(D_3')$ such that, for $|\lambda|$ sufficiently large, we have*

$$|\mathscr{M}v| \leqslant \begin{cases} M|\lambda|^{\frac{2}{3}} & \text{for} \quad z \in \Omega, \\ M|z|^{-1} & \text{for} \quad z \in D_3' \backslash \Omega, \end{cases} \tag{4.1}$$

and

$$v(z_*,\lambda) = c_* + O(\lambda^{-2}), \quad v'(z_*,\lambda) = O(\lambda^{-2}),$$

where c_* is a fixed non-zero constant. Then the equation $\mathscr{M}u = 0$ has a solution $u(z,\lambda)$ in D_3 such that, for $|\lambda|$ sufficiently large, we have

(i) $u(z_*,\lambda) = c_*, \quad u'(z_*,\lambda) = 0,$

(ii) $|u(z,\lambda) - v(z,\lambda)| \leqslant M|\lambda|^{-2}\ln|\lambda|$ *for* $z \in D_3$,

(iii) $|u'(z,\lambda) - v'(z,\lambda)| \leqslant M|\lambda|^{-\frac{4}{3}}$ *for* $z \in D_3$.

Proof. (*a*) We define the *remainder*

$$\rho(z,\lambda) = u(z,\lambda) - v(z,\lambda),$$

so that the problem

$$\left.\begin{aligned} \mathscr{M}u = 0 \quad \text{in} \quad D_3, \\ u(z_*,\lambda) = c_*, \quad u'(z_*,\lambda) = 0, \end{aligned}\right\}$$

becomes

$$\left.\begin{aligned} \mathscr{M}\rho = -\mathscr{M}v \quad \text{in} \quad D_3, \\ \rho(z_*,\lambda) = O(\lambda^{-2}), \quad \rho'(z_*,\lambda) = O(\lambda^{-2}). \end{aligned}\right\} \tag{4.2}$$

(*b*) To solve problem (4.2), we first obtain a fundamental set of solutions $\psi_0, \psi_1, \psi_2, \psi_3$ of the equation $\mathscr{M}\psi = 0$ as follows. Since the equation $\mathscr{N}^*\phi = 0$ is equivalent to the first order system (2.7): $\boldsymbol{\phi}' = \boldsymbol{N}^*\boldsymbol{\phi}$, it is clear from theorem 2.5 that the matrix

$$\boldsymbol{\Phi} = \begin{bmatrix} \phi_0 & \phi_1 & \phi_2 & \phi_3 \\ \phi_0' & \phi_1' & \phi_2' & \phi_3' \\ \lambda^{-2}\phi_0'' & \lambda^{-2}\phi_1'' & \lambda^{-2}\phi_2'' & \lambda^{-2}\phi_3'' \\ \lambda^{-2}\phi_0''' & \lambda^{-2}\phi_1''' & \lambda^{-2}\phi_2''' & \lambda^{-2}\phi_3''' \end{bmatrix}$$

is a fundamental matrix solution of $\boldsymbol{\phi}' = \boldsymbol{N}^*\boldsymbol{\phi}$. Then, by theorem 2.2, it follows that each column $\boldsymbol{\psi}_k$, $(k = 0, 1, 2, 3)$, of the matrix $\boldsymbol{\Psi} = \boldsymbol{G}^{-1}\boldsymbol{\Phi}$ corresponds to one of the linearly independent solutions $\psi_0, \psi_1, \psi_2, \psi_3$ of $\mathscr{M}\psi = 0$.

(*c*) Let the Wronskian of the functions $\psi_0, \psi_1, \psi_2, \psi_3$ be $W \equiv W(\psi_0, \psi_1, \psi_2, \psi_3)$ (which is independent of z), and let $\Delta_k(z,\lambda)$, $k = 0, 1, 2, 3$, be the determinant that results from replacing the column $\psi_k, \ldots, \psi_k'''$ in W by 0, 0, 0, 1. Write $-\mathscr{M}v = f(z,\lambda)$; then the differential equation $\mathscr{M}\rho = f$ in (4.2) is solved by

$$\rho(z,\lambda) = c_0(\lambda)\,\psi_0(z,\lambda) + c_3(\lambda)\,\psi_3(z,\lambda) + W^{-1}\sum_{k=0}^{3}\psi_k(z,\lambda)\int_{z_k}^{z}\Delta_k(t,\lambda)f(t,\lambda)\,\mathrm{d}t. \tag{4.3}$$

Here the coefficients $c_0(\lambda)$ and $c_3(\lambda)$ will be determined later from the boundary conditions, and

$$z_0 = z_*, \quad z_1 = z_l, \quad z_2 = z_r, \quad z_3 = z_*,$$

with the points z_*, z_r and z_l defined by (3.2), (3.3) and (3.4).

(*d*) The next step is to bound the integrals

$$I_k \equiv \left| W^{-1}\psi_k(z,\lambda) \int_{z_k}^{z} \Delta_k(t,\lambda) f(t,\lambda)\,\mathrm{d}t \right|, \tag{4.4}$$

and

$$J_k \equiv \left| W^{-1}\psi_k'(z,\lambda) \int_{z_k}^{z} \Delta_k(t,\lambda) f(t,\lambda)\,\mathrm{d}t \right|. \tag{4.5}$$

Recall the transformation $\xi = \frac{2}{3}\mathrm{i}\lambda z^{\frac{3}{2}}$ of § 3, and change the variable of integration in the integrals in (4.4) and (4.5) to η, where

$$\eta = \tfrac{2}{3}\,\mathrm{i}\lambda t^{\frac{3}{2}}.$$

In order to use lemmas 3.1 and 3.2, we bound the right-hand sides of (4.4) and (4.5) in terms of ξ, η and λ as follows. First, recall that the matrix $\boldsymbol{G}^{-1}$ in $\boldsymbol{\Psi} = \boldsymbol{G}^{-1}\boldsymbol{\Phi}$ belongs to Λ. Moreover, if we explicitly invert the matrix $\boldsymbol{G}_0$ as given by (2.21), we find for $\boldsymbol{G}_0^{-1} = (g_{ij})$ that

$$\left.\begin{aligned} &g_{12}(z) = O(z^2), \quad g_{14} = O(z) \quad \text{for} \quad |z| \to 0, \\ \text{and} \qquad &g_{31} = g_{32} = g_{34} = g_{41} = g_{42} = 0, \end{aligned}\right\}$$

where we have also used (2.33). Then, from the bounds (2.65) of the elements of $\boldsymbol{\Phi}$, we deduce the following estimates for the functions $(\mathrm{d}/\mathrm{d}z)^j\,\psi_k$ $(j = 0, 1, 2, 3)$.

$$\left.\begin{aligned} &\psi_0 = \lambda^{-\frac{2}{3}}(\operatorname{tru}|\xi|^{\frac{2}{3}})\,E(z,\lambda), \quad (\mathrm{d}/\mathrm{d}z)^j\,\psi_0 = E(z,\lambda) \quad (j = 1,2,3); \\ &\begin{cases} \psi_1 = \{\lambda^{-\frac{2}{3}}(\operatorname{tru}|\xi|^{-\frac{5}{6}}) + \lambda^{-\frac{4}{3}}(\operatorname{tru}|\xi|^{\frac{5}{6}})\}\,\mathrm{e}^{-\xi}E(z,\lambda), \\ (\mathrm{d}/\mathrm{d}z)^j\,\psi_1 = \lambda^{\frac{2}{3}(j-1)}(\operatorname{tru}|\xi|^{\frac{1}{3}j-\frac{5}{6}})\,e^{-\xi}E(z,\lambda) \quad (j = 1,2,3); \end{cases} \\ &\begin{cases} \psi_2 = \{\lambda^{-\frac{2}{3}}(\operatorname{tru}|\xi|^{-\frac{5}{6}}) + \lambda^{-\frac{4}{3}}(\operatorname{tru}|\xi|^{\frac{5}{6}})\}\,\mathrm{e}^{\xi}E(z,\lambda), \\ (\mathrm{d}/\mathrm{d}z)^j\,\psi_2 = \lambda^{\frac{2}{3}(j-1)}(\operatorname{tru}|\xi|^{\frac{1}{3}j-\frac{5}{6}})\,\mathrm{e}^{\xi}E(z,\lambda) \quad (j = 1,2,3); \end{cases} \\ &\begin{cases} \psi_3 = E(z,\lambda), \quad \mathrm{d}\psi_3/\mathrm{d}z = \ln\{\lambda^{-\frac{2}{3}}(\operatorname{tru}|\xi|^{\frac{2}{3}})\}\,E(z,\lambda), \\ (\mathrm{d}/\mathrm{d}z)^j\,\psi_3 = \lambda^{\frac{2}{3}(j-1)}(\operatorname{tru}|\xi|^{-\frac{2}{3}(j-1)})\,E(z,\lambda) \quad (j = 2,3). \end{cases} \end{aligned}\right\} \tag{4.6}$$

Now, using these bounds, we can show that

$$\left.\begin{aligned} \Delta_0(t,\lambda) &= \lambda^{\frac{2}{3}}(\operatorname{tru}|\eta|^{-\frac{2}{3}})\,E(t,\lambda), \\ \Delta_1(t,\lambda) &= \lambda^{\frac{2}{3}}(\operatorname{tru}|\eta|^{-\frac{1}{6}})\,\mathrm{e}^{\eta}E(t,\lambda), \\ \Delta_2(t,\lambda) &= \lambda^{\frac{2}{3}}(\operatorname{tru}|\eta|^{-\frac{1}{6}})\,\mathrm{e}^{-\eta}E(t,\lambda), \\ \Delta_3(t,\lambda) &= E(t,\lambda). \end{aligned}\right\} \tag{4.7}$$

Next, it follows from (4.1) that

$$|f(t,\lambda)| = |\mathscr{M}v| \leqslant M|\lambda|^{\frac{2}{3}}\operatorname{tru}|\eta|^{-\frac{2}{3}}; \tag{4.8}$$

and from (2.67), since $\det \boldsymbol{G}_0^{-1} = 1$, that

$$|W^{-1}| \leqslant M|\lambda|^{-2}. \tag{4.9}$$

By virtue of the estimates (4.6) to (4.9), and lemmas 3.1 and 3.2, we now have that

$$I_0 \leqslant M|\lambda|^{-2}(\operatorname{tru}|\xi|^{\frac{2}{3}}) \int_{\xi_*}^{\xi} |(\operatorname{tru}|\eta|^{-\frac{4}{3}})\,\eta^{-\frac{1}{3}}\,\mathrm{d}\eta| \leqslant M|\lambda|^{-2},$$

and, similarly,

$$I_1 \leqslant M|\lambda|^{-2}\{\text{tru}\,|\xi|^{-2}+|\lambda|^{-\frac{2}{3}}\,(\text{tru}\,|\xi|^{-\frac{1}{3}})\},$$

$$I_2 \leqslant M|\lambda|^{-2}\{\text{tru}\,|\xi|^{-2}+|\lambda|^{-\frac{2}{3}}(\text{tru}\,|\xi|^{-\frac{1}{3}})\},$$

$$I_3 \leqslant M|\lambda|^{-2}\ln|\lambda|.$$

Also,

$$J_0 \leqslant M|\lambda|^{-\frac{4}{3}},$$

$$J_1 \leqslant M|\lambda|^{-\frac{4}{3}}\,(\text{tru}\,|\xi|^{-\frac{5}{3}}),$$

$$J_2 \leqslant M|\lambda|^{-\frac{4}{3}}\,(\text{tru}\,|\xi|^{-\frac{5}{3}}),$$

$$J_3 \leqslant M|\lambda|^{-2}\ln^2|\lambda|.$$

These estimates for I_k, J_k hold for $\xi\in\mathscr{D}$. Since they can easily be shown to hold also for those points $z\in\Omega$ which are not contained in the z-image of $W\,(\subset\mathscr{D})$, we deduce that, for $|\lambda|$ sufficiently large,

$$\sum_{k=0}^{3} I_k \leqslant M|\lambda|^{-2}\ln|\lambda| \quad \text{for} \quad z\in D_3,$$

and

$$\sum_{k=0}^{3} J_k \leqslant M|\lambda|^{-\frac{4}{3}} \quad \text{for} \quad z\in D_3.$$

(*e*) With these bounds on the ψ_1 and ψ_2 parts of ρ in (4.3), the boundary conditions become

$$\left.\begin{aligned} c_0(\lambda)\,\psi_0(z_*,\lambda)+c_3(\lambda)\,\psi_3(z_*,\lambda) &= O(\lambda^{-2}),\\ c_0(\lambda)\,\psi_0'(z_*,\lambda)+c_3(\lambda)\,\psi_3'(z_*,\lambda) &= O(\lambda^{-2}),\end{aligned}\right\} \tag{4.10}$$

and if we can show that the determinant

$$\Delta_* = \begin{vmatrix} \psi_0(z_*,\lambda) & \psi_3(z_*,\lambda) \\ \psi_0'(z_*,\lambda) & \psi_3'(z_*,\lambda) \end{vmatrix}$$

satisfies $|\Delta_*| \geqslant k > 0$ for some k independent of λ, it will follow that c_0 and c_3 are $O(\lambda^{-2})$. Now

$$\Delta_* = \Delta_{\mathrm{I}}\Delta_{\mathrm{II}} + O(\lambda^{-2}\ln\lambda),$$

where

$$\Delta_{\mathrm{I}} = \begin{vmatrix} g_{11}(z_*) & g_{12}(z_*) \\ g_{21}(z_*) & g_{22}(z_*) \end{vmatrix} \quad \text{and} \quad \Delta_{\mathrm{II}} = \begin{vmatrix} \phi_{0,0}(z_*) & \phi_{3,0}(z_*) \\ \phi_{0,0}'(z_*) & \phi_{3,0}'(z_*) \end{vmatrix},$$

and (because $g_{31}=g_{32}=g_{41}=g_{42}=0$ and $g_{34}=0$)

$$\Delta_{\mathrm{I}} = \frac{\det \mathbf{G}_0^{-1}}{g_{33}\,g_{44}} = \frac{1}{g_{33}\,g_{44}}.$$

Since $|g_{33}|$ and $|g_{44}|$ are bounded above, at z_*, $|\Delta_{\mathrm{I}}|$ is bounded below. The Wronskian Δ_{II} is also independent of λ, and is not zero because $\phi_{0,0}$ and $\phi_{3,0}$ are linearly independent solutions of

$$z\phi''+\beta_0\phi = 0.$$

(*f*) With $c_0(\lambda)$ and $c_3(\lambda)$ chosen to satisfy (4.10), the function $\rho(z,\lambda)$ of (4.3) is a solution of problem (4.2), and $u=v+\rho$ has the properties claimed in the theorem.

In our final application (§6) we shall need to know the asymptotic behaviour of the solution $u(z,\lambda)$ just established to *arbitrary order* in λ, when z lies in a neighbourhood of the point z_*. We therefore prove

THEOREM 4.2. *Let v and u be as in theorem 4.1. Suppose further that* $|v(z,\lambda)| \leqslant M$ *for* $z \in D_3$ *and* $|\lambda| \geqslant |\lambda_0|$, *with* $|\lambda_0|$ *sufficiently large.*

Then $|u(z,\lambda)| \leqslant M$ *for* $z \in D_3$ *and* $|\lambda| \geqslant |\lambda_0|$; *and, in a fixed neighbourhood* N_* *of* z_*, *we have*

$$u(z,\lambda) = \sum_{n=0}^{m} \lambda^{-2n} u_{2n}(z) + R_m(z,\lambda),$$

where all the functions u_{2n} *belong to* $\mathscr{H}(N_*)$, *where m is an arbitrary non-negative integer, and where*

$$(\mathrm{d}/\mathrm{d}z)^j R_m(z,\lambda) = O(\lambda^{-2m-2}\ln\lambda) \quad \textit{for} \quad |\lambda| \to \infty, z \in N_* \quad (j = 0, \ldots, 4).$$

Proof. (*a*) From the hypothesis $|v(z,\lambda)| \leqslant M$, and theorem 4.1, it is clear that $|u(z,\lambda)| \leqslant M$ for $z \in D_3$ and $|\lambda| \geqslant |\lambda_0|$, with $|\lambda_0|$ sufficiently large.

(*b*) Consider the fundamental set of solutions ψ_0, ψ_1, ψ_2, ψ_3 of $\mathscr{M}\psi = 0$ which were obtained from the transformation $\boldsymbol{\Psi} = \boldsymbol{G}^{-1}\boldsymbol{\Phi}$ in theorem 4.1. Then clearly there exist coefficients $k_0(\lambda)$, $k_1(\lambda)$, $k_2(\lambda)$, $k_3(\lambda)$ such that

$$u(z,\lambda) = k_0(\lambda)\,\psi_0(z,\lambda) + k_1(\lambda)\,\psi_1(z,\lambda) + k_2(\lambda)\,\psi_2(z,\lambda) + k_3(\lambda)\,\psi_3(z,\lambda). \tag{4.11}$$

But, using (2.64*b*), we can show that

$$\left.\begin{aligned}\psi_1 &\sim \text{const.} \times \lambda^{-\frac{1}{2}}\mathrm{e}^{-\xi}\\ \psi_2 &\sim \text{const.} \times \lambda^{-\frac{1}{2}}\mathrm{e}^{\xi}\end{aligned}\right\} \quad \text{for } |\lambda| \to \infty \quad \text{and fixed} \quad z \in D_3\backslash\{0\}.$$

Hence

$$k_1(\lambda) = O(\lambda^{\frac{1}{2}}\exp\{-K_2\cos(\tfrac{1}{4}\pi - \delta_2)\,|\lambda|\}), \quad k_2(\lambda) = O(\lambda^{\frac{1}{2}}\exp\{-K_1\cos(\tfrac{1}{4}\pi - \delta_2)\,|\lambda|\}), \tag{4.12}$$

where the positive numbers K_1, K_2 and $\delta_2 < \frac{1}{4}\pi$ are those occurring in the definitions (3.7) of the sets $T_1, T_2 \subset \mathscr{D}$ in the ξ-plane. For suppose that (4.12) is not true. Then, if we consider points z_{I} and z_{II} defined by

$$|z_{\mathrm{I}}| = R_1, \quad \arg z_{\mathrm{I}} = -\delta_1,$$
$$|z_{\mathrm{II}}| = R_2, \quad \arg z_{\mathrm{II}} = \pi + \delta_1,$$

and recall that $\psi_0(z,\lambda)$ and $\psi_3(z,\lambda)$ do not grow exponentially, we get a contradiction of

$$|u(z,\lambda)| \leqslant M$$

upon evaluating $u(z,\lambda)$ from (4.11) at points $z \in D_3$ which are sufficiently near z_{I}, and at points $z \in D_3$ which are sufficiently near z_{II}.

It is now clear that if we define a fixed point z_0 by

$$z_0 \in S_1, \quad |z_0| > |z_*|, \quad \arg z_0 = \arg z_*,$$

and let N_* be the intersection of S_1 with a disk about z_* of radius

$$k_* = \min\{\tfrac{1}{2}|z_* - z_0|, \tfrac{1}{2}|z_*|\},$$

then
$$u(z,\lambda) = k_0(\lambda)\,\psi_0(z,\lambda) + k_3(\lambda)\,\psi_3(z,\lambda) + g(z,\lambda) \quad \text{in} \quad N_*, \tag{4.13}$$

where the functions $(\mathrm{d}/\mathrm{d}z)^j g(z,\lambda)$ $(j = 0, \ldots, 4)$ are exponentially small in N_*.

(*c*) The boundary conditions satisfied by $u(z,\lambda)$ now become

$$\left.\begin{aligned}k_0(\lambda)\,\psi_0(z_*,\lambda) + k_3(\lambda)\,\psi_3(z_*,\lambda) &= c_* - g(z_*,\lambda),\\ k_0(\lambda)\,\psi_0'(z_*,\lambda) + k_3(\lambda)\,\psi_3'(z_*,\lambda) &= -g'(z_*,\lambda).\end{aligned}\right\}$$

This system is analogous to (4.10), and similar arguments apply. In fact, using the result (2.66) of theorem 2.5, we can now show from (4.13) that $u(z,\lambda)$ has the properties stated in the theorem.

5. An exponential solution

THEOREM 5.1. *Suppose that there exists a function* $w(.,\lambda)\in\mathscr{H}(D_3')$ *such that, in* D_3', *we have*

$$|\mathscr{M}w| \leqslant \begin{cases} M|\lambda|^2 & \text{for} \quad z\in\Omega, \\ M|\lambda|^{\frac{13}{6}}|z|^{\frac{1}{4}}|\mathrm{e}^{-\xi}| & \text{for} \quad |z|\leqslant|\lambda|^{-\frac{1}{3}}, z\notin\Omega, \\ M|\lambda|^{\frac{1}{6}}|z|^{-\frac{15}{4}}|\mathrm{e}^{-\xi}| & \text{for} \quad |z|\geqslant|\lambda|^{-\frac{1}{3}}, \end{cases} \tag{5.1}$$

for sufficiently large $|\lambda|$, *and* (*for* $j=0,1,2,3$)

$$(\mathrm{d}/\mathrm{d}z)^j w|_{z=z_*} = o(\lambda^{-n}) \quad as \quad |\lambda|\to\infty \quad for\ every \quad n. \tag{5.2}$$

Then the equation $\mathscr{M}u=0$ *has a solution* $u(z,\lambda)$ *in* D_3 *such that* (*for* $j=0,1,2,3$)

$$\text{(i)} \quad (\mathrm{d}/\mathrm{d}z)^j u|_{z=z_*} = o(\lambda^{-n}) \quad as \quad |\lambda|\to\infty \quad for\ every \quad n, \tag{5.3}$$

$$\text{(ii)} \quad |u(z,\lambda)-w(z,\lambda)| \leqslant \begin{cases} M|\lambda|^{-\frac{1}{2}} & \text{for} \quad z\in\Omega, \\ M|\lambda|^{-\frac{4}{3}}|z|^{-\frac{5}{4}}|\mathrm{e}^{-\xi}| & \text{for} \quad z\in D_3\backslash\Omega, \end{cases} \tag{5.4}$$

$$\text{(iii)} \quad |u'(z,\lambda)-w'(z,\lambda)| \leqslant \begin{cases} M|\lambda|^{\frac{1}{6}} & \text{for} \quad z\in\Omega, \\ M|\lambda|^{-\frac{11}{6}}|z|^{-\frac{15}{4}}|\mathrm{e}^{-\xi}| & \text{for} \quad z\in D_3\backslash\Omega, \end{cases} \tag{5.5}$$

for $|\lambda|$ *sufficiently large.*

Proof. (*a*) Define the remainder $\rho(z,\lambda)=u(z,\lambda)-w(z,\lambda)$, so that the equation $\mathscr{M}u=0$ becomes

$$\mathscr{M}\rho = -\mathscr{M}w \equiv h(z,\lambda), \quad \text{say.}$$

Then, (cf. (4.3)),

$$\rho(z,\lambda) = W^{-1}\sum_{k=0}^{3}\psi_k(z,\lambda)\int_{z_k}^{z}\Delta_k(t,\lambda)\,h(t,\lambda)\,\mathrm{d}t \tag{5.6}$$

satisfies $\mathscr{M}\rho=h$. Here the symbols W, ψ_k and Δ_k have the same meaning as before, but we now choose

$$z_0=z_r, \quad z_1=z_*, \quad z_2=z_r, \quad z_3=z_r.$$

(*b*) We write

$$\left.\begin{aligned} I_k &\equiv \left|W^{-1}\sum_{k=0}^{3}\psi_k(z,\lambda)\int_{z_k}^{z}\Delta_k(t,\lambda)\,h(t,\lambda)\,\mathrm{d}t\right|, \\ J_k &\equiv \left|W^{-1}\sum_{k=0}^{3}\psi_k'(z,\lambda)\int_{z_k}^{z}\Delta_k(t,\lambda)\,h(t,\lambda)\,\mathrm{d}t\right|, \end{aligned}\right\} \tag{5.7}$$

and, as before, change the variable of integration to $\eta=(\frac{2}{3})\,\mathrm{i}\lambda t^{\frac{3}{2}}$. Next, if we compare the estimates (5.1) of the function $h(z,\lambda)$ ($\equiv -\mathscr{M}w$) with the estimates (3.11) of the function $F(\xi,\lambda)$ introduced in § 3 (*c*), we find that

$$|h(t,\lambda)| \leqslant M|\lambda|^2|F(\eta,\lambda)|\,|\mathrm{e}^{-\eta}| \quad \text{for} \quad \eta\in\mathscr{D}'.$$

This estimate, together with the bounds (4.6) to (4.9), and Lemmas 3.3 and 3.5, now imply the following estimates for the integrals I_k, J_k.

$$I_0 \leqslant M|\lambda|^{-\frac{2}{3}}\,(\operatorname{tru}|\xi|^{\frac{2}{3}})\int_{\xi}^{\xi_r}|(\operatorname{tru}|\eta|^{-\frac{2}{3}})\,\eta^{-\frac{1}{3}}F(\eta,\lambda)\,\mathrm{e}^{-\eta}\mathrm{d}\eta|$$

$$\leqslant \begin{cases} M|\lambda|^{-\frac{2}{3}}(\operatorname{tru}|\xi|^{-\frac{1}{6}})\,|\mathrm{e}^{-\xi}| & \text{for} \quad |\xi|\leqslant|\lambda|^{\frac{1}{4}}, \\ M|\xi|^{-\frac{17}{6}}|\mathrm{e}^{-\xi}| & \text{for} \quad |\xi|\geqslant|\lambda|^{\frac{1}{4}}; \end{cases}$$

and, similarly,

$$I_k \leqslant \begin{cases} Mf_k(\xi,\lambda)|\mathrm{e}^{-\xi}| & \text{for} \quad |\xi|\leqslant|\lambda|^{\frac{1}{4}} \\ Mg_k(\xi,\lambda)|\mathrm{e}^{-\xi}| & \text{for} \quad |\xi|\geqslant|\lambda|^{\frac{1}{4}} \end{cases} \quad (k=1,2,3),$$

where

$$\left.\begin{aligned} f_1 &= |\lambda|^{-\frac{1}{2}}\,\mathrm{tru}\,|\xi|^{-\frac{5}{6}}, & g_1 &= |\xi|^{-\frac{17}{6}}+|\lambda|^{-\frac{2}{3}}|\xi|^{-\frac{7}{6}},\\ f_2 &= |\lambda|^{-\frac{2}{3}}\,\mathrm{tru}\,|\xi|^{-\frac{7}{6}}, & g_2 &= |\xi|^{-\frac{23}{6}}+|\lambda|^{-\frac{2}{3}}|\xi|^{-\frac{13}{6}},\\ f_3 &= |\lambda|^{-\frac{2}{3}}\,\mathrm{tru}\,|\xi|^{-\frac{1}{6}}, & g_3 &= |\xi|^{-\frac{17}{6}}. \end{aligned}\right\}$$

Also,

$$J_k \leqslant \left\{\begin{matrix} Mp_k(\xi,\lambda)\,|\mathrm{e}^{-\xi}| & \text{for} & |\xi| \leqslant |\lambda|^{\frac{1}{4}} \\ Mq_k(\xi,\lambda)\,|\mathrm{e}^{-\xi}| & \text{for} & |\xi| \geqslant |\lambda|^{\frac{1}{4}} \end{matrix}\right\} \quad (k = 0, 1, 2, 3),$$

where

$$\left.\begin{aligned} p_0 &= \mathrm{tru}\,|\xi|^{-\frac{5}{6}}, & q_0 &= |\lambda|^{\frac{2}{3}}|\xi|^{-\frac{7}{2}},\\ p_1 &= |\lambda|^{\frac{1}{6}}\,\mathrm{tru}\,|\xi|^{-\frac{1}{2}}, & q_1 &= |\lambda|^{\frac{2}{3}}|\xi|^{-\frac{5}{2}},\\ p_2 &= \mathrm{tru}\,|\xi|^{-\frac{5}{6}}, & q_2 &= |\lambda|^{\frac{2}{3}}|\xi|^{-\frac{7}{2}},\\ p_3 &= |\lambda|^{-\frac{2}{3}}\ln|\lambda|\,(\mathrm{tru}|\xi|^{-\frac{1}{6}}), & q_3 &= \ln|\lambda|\quad|\xi|^{-\frac{17}{6}}. \end{aligned}\right\}$$

(*c*) The estimates above, together with (5.6), (5.7), and the fact that $\rho = u-w$, yield the results (5.4) and (5.5) of the theorem. The result (5.3) is a consequence of the hypothesis (5.2) and the fact that ρ and its derivatives are exponentially small at $z = z_*$.

6. Justification of Eagles's approximations

(*a*) *The Orr–Sommerfeld equation*

We now consider the Orr–Sommerfeld differential equation

$$\frac{\mathrm{d}^4\Phi}{\mathrm{d}y^4}-2k^2\frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2}+k^4\Phi-\mathrm{i}kR\left\{(w(y)-c)\left(\frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2}-k^2\Phi\right)-w''(y)\,\Phi\right\}=0$$

in a complex neighbourhood of $[0, 1]$, and study its asymptotic solutions for large kR. We assume that the function $w(y)$ enjoys the following properties:

(*a*) $w(y)$ is real for $y\in[0, 1]$,

(*b*) derivatives of odd order vanish at $y = 1$: $w^{(2n+1)}(1) = 0$,

(*c*) $w(y)$ is a holomorphic function of y in some fixed complex domain D containing the real interval $[0, 1]$,

(*d*) $w(0) = 0$,

(*e*) $w'(y) > 0$ for $y\in[0, 1)$,

(*f*) for given c, with $\mathrm{Im}\,(c)$ sufficiently small, there exists exactly one turning point $y_c\in D$ such that (i) $w(y_c) = c$, and (ii) y_c tends to a point of $(0, 1)$ as $\mathrm{Im}\,(c)\to 0$.

Now introduce the variable

$$s = y-y_c,$$

define the function $V(s) = w(s+y_c)-w(y_c)$, and let the real closed interval $[s_a, s_b]$, with $s_a < 0 < s_b$, be such that $s_a = -|y_c|$, $s_b = |1-y_c|$. The hypotheses (*a*) to (*f*) above then imply the existence of a domain D_1 in the s-plane such that (for $\mathrm{Im}\,(c)$ sufficiently small):

(*a*) $-y_c, 1-y_c\in D_1$,

(*b*) $[s_a, s_b]\subset D_1$,

(*c*) $V\in\mathscr{H}(D_1)$,

(*d*) $V(0) = 0, \quad V'(0) \neq 0$,

(*e*) $V(s) \neq 0 \quad \text{for} \quad s\in D_1\backslash\{0\}$,

(*f*) $\int_0^s \sqrt{V(s')}\,\mathrm{d}s' \neq 0 \quad \text{for} \quad s\in D_1\backslash\{0\}$.

Consequently, if we write w_c' for $w'(y_c)$, and let

$$\lambda^2 = -\mathrm{i}kRw_c', \quad 0 \leqslant \arg w_c' < 2\pi,$$

$$P(s,\lambda) = V(s)/w_c' + \lambda^{-2}(-2k^2) \equiv P_0(s) + \lambda^{-2}P_2(s),$$

and $$R(s,\lambda) = \{-V''(s) - k^2V(s)\}/w_c' + \lambda^{-2}k^4 \equiv R_0(s) + \lambda^{-2}R_2(s),$$

the Orr–Sommerfeld equation becomes

$$\mathscr{L}\Phi \equiv \frac{\mathrm{d}^4\Phi}{\mathrm{d}s^4} + \lambda^2\left\{P(s,\lambda)\frac{\mathrm{d}^2\Phi}{\mathrm{d}s^2} + R(s,\lambda)\,\Phi\right\} = 0 \quad \text{in} \quad D_1. \tag{6.1}$$

This is a particular case of the equation investigated in § 2 ; henceforth $\mathscr{L}$ denotes the operator in (6.1).

(*b*) *The formal approximations of Eagles*

Eagles (1969) defined $$\epsilon = (kRw_c')^{-\frac{1}{3}} = (\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\lambda)^{-\frac{2}{3}},$$

so that $$\lambda = \mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\epsilon^{-\frac{3}{2}}.$$

He introduced the *stretching transformation*† $s = \epsilon\sigma$, and for bounded values of σ (that is, for an *inner* region), wrote the equation $\mathscr{L}\Phi = 0$ in the form

$$\frac{\mathrm{d}^4\Phi}{\mathrm{d}\sigma^4} - \mathrm{i}\sigma\frac{\mathrm{d}^2\Phi}{\mathrm{d}\sigma^2} = \mathrm{i}H\epsilon\left(\tfrac{1}{2}\sigma^2\frac{\mathrm{d}^2\Phi}{\mathrm{d}\sigma^2} - \Phi\right) + O(\epsilon^2),$$

where $$H \equiv w''(y_c)/w_c' \equiv w_c''/w_c',$$

and where the term implied by the O-symbol is known explicitly. He then defined exact solutions Φ_1, Φ_2 and Φ_3 of $\mathscr{L}\Phi = 0$ by means of convergent series

$$\Phi_j = \sum_{n=0}^{\infty} \epsilon^n \chi_j^{(n)}(\sigma) \quad (j = 1, 2, 3), \tag{6.2}$$

and computed the first few terms of these series. These solutions have the following properties.

(*a*) The functions $\chi_1^{(n)}$, $\chi_2^{(n)}$ $(n = 0, 1, 2, \ldots)$ increase only *algebraically* (and not exponentially) as $|\sigma| \to \infty$ with $-\frac{7}{6}\pi + \delta \leqslant \arg\sigma \leqslant \frac{1}{6}\pi - \delta$ (for some fixed $\delta > 0$).

(*b*) Every function $\chi_3^{(n)}$ has a factor $\exp\left(-\left(\frac{2}{3}\right)\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\sigma^{\frac{3}{2}}\right)$ for large $|\sigma|$ and

$$-\tfrac{7}{6}\pi + \delta \leqslant \arg\sigma \leqslant \tfrac{5}{6}\pi - \delta;$$

hence the $\chi_3^{(n)}(n = 0, 1, 2, \ldots)$ *decrease exponentially* as $|\sigma| \to \infty$ with $-\frac{1}{2}\pi + \delta \leqslant \arg\sigma \leqslant \frac{1}{6}\pi - \delta$.

Next, Eagles constructed formal asymptotic series ϕ_5 and ϕ_7, which were expected to approximate solutions of $\mathscr{L}\Phi = 0$ under the *outer* limit: $|\epsilon| \to 0$ with $|s|$ bounded away from zero. The first of these,

$$\phi_5 = \sum_{n=0}^{\infty} (w_c'\epsilon)^{3n}\,\phi_5^{(n)}(y), \tag{6.3}$$

was found by formal substitution into the Orr–Sommerfeld equation written in the form

$$\{w(y) - c\}\left\{\frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2} - k^2\Phi\right\} - w''(y)\,\Phi = \frac{\mathrm{i}}{w_c'}\epsilon^3\left\{\frac{\mathrm{d}^4\Phi}{\mathrm{d}y^4} - 2k^2\frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2} + k^4\Phi\right\},$$

with $$\phi_5|_{y=1} = w(1) - c, \quad \frac{\mathrm{d}}{\mathrm{d}y}\phi_5|_{y=1} = 0, \quad \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^3\phi_5|_{y=1} = 0. \tag{6.4}$$

† Our variables s, σ correspond to the variables z, η used by Eagles (1969).

In contrast to ϕ_5, which is *bounded*, the asymptotic series ϕ_7 is *unbounded*, and has the form

$$\phi_7 = \exp\left\{-\epsilon^{-\frac{3}{2}}(w_c')^{-\frac{1}{2}}\int_0^s \sqrt{\{\mathrm{i}V(s')\}}\,\mathrm{d}s'\right\}\{V^{-\frac{5}{4}}(s)+O(\epsilon^{\frac{3}{2}})\}.$$

Eagles then used the method of 'matched asymptotic expansions' (Fraenkel 1969) to construct (*a*) from the 'inner' solutions Φ_1, Φ_2 and the formal 'outer' series ϕ_5, *a function that he called* $E_{4,4}^{\text{comp}}\Phi_5$ *and that we shall call* $\tilde{v}$;

(*b*) from the 'inner' solution Φ_3 and the formal 'outer' series ϕ_7 *a function that he called*

$$G\hat{E}_{0,0}^{\text{comp}}(\Phi_3/G)$$

and that we shall call $\tilde{w}$. Here

$$G(s,\epsilon) = \sigma^{-\frac{5}{4}}\exp\left\{-\epsilon^{-\frac{3}{2}}(w_c')^{-\frac{1}{2}}\int_0^s \sqrt{\{\mathrm{i}V(s')\}}\,\mathrm{d}s'\right\}. \tag{6.5}$$

(Some details of $\tilde{v}$ and $\tilde{w}$ are given in appendix B.) Eagles then assumed the following on the basis of the first few terms of $\Phi_1, \ldots, \phi_7$.

(A) There exists a solution Φ_5 of $\mathscr{L}\Phi = 0$ such that

(i) $\Phi_5 - \tilde{v} = o(\epsilon^2)$,

(ii) $\Phi_5|_{s=1-y_c} = V(1-y_c)$, $\quad(\mathrm{d}/\mathrm{d}s)\,\Phi_5|_{s=1-y_c} = 0$,

(iii) $(\mathrm{d}/\mathrm{d}s)^3\,\Phi_5|_{s=1-y_c} = o(\epsilon^n)$ for every n,

(iv) $\Phi_5 = O(1)$.

(B) There exists a solution Φ_3 of $\mathscr{L}\Phi = 0$ such that

(i) $\Phi_3 - \chi_3^{(0)} = o(1)$ for $|s| \preccurlyeq |\epsilon|$,†

$\Phi_3 - \tilde{w} = G(s,\epsilon)\,\rho(s,\epsilon)$ for $|s| \succ |\epsilon|$, with $\rho(s,\epsilon) = o(1)$;

(ii) $(\mathrm{d}/\mathrm{d}s)\,\Phi_3|_{s=1-y_c} = o(\epsilon^n)$, $\quad(\mathrm{d}/\mathrm{d}s)^3\,\Phi_3|_{s=1-y_c} = o(\epsilon^n)$ for every n.

(*c*) *The modified approximations v and w*

The formal approximations $\tilde{v}$ and $\tilde{w}$ are single-valued only in a *cut* s-plane, where the cut extends from the origin to infinity in a certain excluded sector (Eagles 1969, Figure 2). Despite this, $\tilde{v}$ and $\tilde{w}$ were adequate for Eagles's (1969) final applications, in which only the functions and their *first* derivatives were important.

To justify these approximations by means of the foregoing theorems, we must first construct functions v and w *whose second, third and fourth derivatives are bounded in a neighbourhood of the origin* (where these higher derivatives of Eagles's $\tilde{v}$ and $\tilde{w}$ are singular) *but which, with their first derivatives, do not differ significantly from Eagles's functions* throughout their domain of definition. For then $\mathscr{L}v$ and $\mathscr{L}w$ can be bounded uniformly, which is not possible for $\mathscr{L}\tilde{v}$ and $\mathscr{L}\tilde{w}$.

Let Ω_0 be the disk

$$\Omega_0 = \{s\,|\,|s| < K_0|\epsilon|\}, \tag{6.6}$$

where K_0 is some fixed (large) positive number; Ω_0 characterizes the inner region because $s = \epsilon\sigma$. The following lemma is crucial for the construction of v and w: we seek a function $u(\sigma)$ which behaves like σ to all orders for $|\sigma| \to \infty$ in the cut plane

$$C' = C\backslash\{\sigma|\sigma = \mathrm{i}\eta, \quad \eta \geqslant 2K_0\},$$

but which is bounded away from zero in C', for then we can replace $\ln\sigma$ in the function $\tilde{v}$ by a function $\ln u(\sigma)$.

† Following Fraenkel (1969), we use Hardy's notation for relative orders of magnitude. If $f(\epsilon) = o\{g(\epsilon)\}$ for $|\epsilon| \to 0$, we write $f \prec g$ or $g \succ f$. If $f = O(g)$, we write $f \preccurlyeq g$.

LEMMA 6.1. *The conformal transformation* (figure 8)

$$\tau^4 = \tfrac{1}{2}e^{\frac{1}{2}\pi i}\left(\frac{\sigma}{K_0} - 2i\right), \quad -\tfrac{1}{4}\pi < \arg\tau < \tfrac{1}{4}\pi, \quad -\tfrac{3}{2}\pi < \arg\left(\frac{\sigma}{K_0} - 2i\right) < \tfrac{1}{2}\pi, \tag{6.7}$$

maps the cut plane $\mathbf{C}'$ *on to the sector*

$$S = \{\tau \mid -\tfrac{1}{4}\pi < \arg\tau < \tfrac{1}{4}\pi, \quad \tau \neq 0\},$$

and the function

$$u(\sigma) = -2iK_0(\tau^2+1)(\tau+1)(\tau-1+2e^{-\tau}) \tag{6.8}$$

has the following properties:

(i) $u \in \mathscr{H}(\mathbf{C}')$,

(ii) $|u(\sigma)|$ *is bounded away from zero in* $\mathbf{C}'$,

(iii) $u(\sigma) = \sigma + f(\sigma)$,

where

$$f(\sigma) = o(\sigma^{-n}) \quad as \quad |\sigma| \to \infty \quad in \quad \mathbf{C}' \quad for\ every \quad n. \tag{6.9}$$

For the proof see appendix *C*.

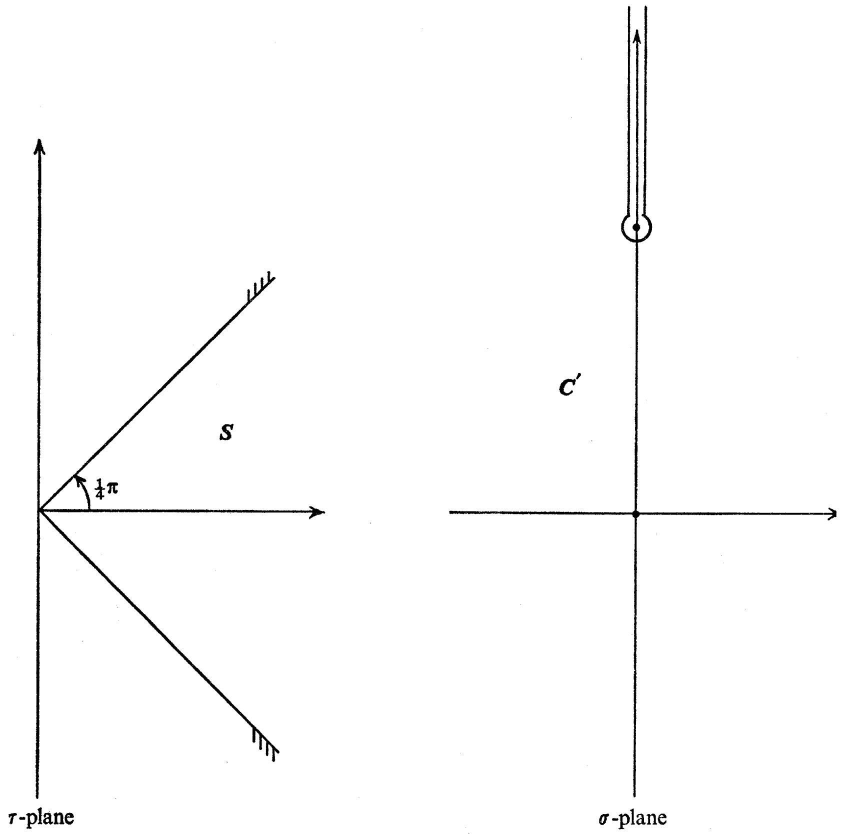

FIGURE 8. The mapping (6.7).

Remark. If, instead of $u(\sigma)$ in (6.8), we consider the (more easily analysed) function

$$\tilde{u}(\sigma) = -2iK_0(\tau^2+1)(\tau+1)(\tau-1+e^{-\tau}), \tag{6.10}$$

then lemma 6.1 holds, except that $\tilde{u}(\sigma)$ has a zero at $\tau = 0$ ($\sigma = 2iK_0$).

Proof of remark. By (6.7) and (6.10),

$$\sigma = -2iK_0(\tau^4-1) = -2iK_0(\tau^2+1)(\tau+1)(\tau-1),$$

and
$$\tilde{u}(\sigma) = \sigma - 2iK_0(\tau^2+1)(\tau+1)\,e^{-\tau},$$

which implies the properties (i) and (iii) of $\tilde{u}(\sigma)$. It remains to prove that $\tilde{u}(\sigma)$ has no zero in C'.

Returning to (6.10), we can easily check that the factors τ^2+1 and $\tau+1$ are bounded away from zero in S. Now consider the factor
$$p(\tau) = \tau - 1 + e^{-\tau}$$
in (6.10). First, write $\tau = x + iy = re^{i\theta}$ and note that

$$\operatorname{Re}\{p'(\tau)\} = 1 - e^{-x}\cos y > 0 \quad \text{for} \quad \tau \in S.$$

Hence, if we write $e^{-i\theta}p(\tau) \equiv g + ih$, it follows that $\partial g/\partial r = \operatorname{Re}\{p'(\tau)\} > 0$ for $\tau \in S$.

But $g = 0$ at $\tau = 0$; hence $g \neq 0$ in S, and $p(\tau) \neq 0$ for $\tau \in S$.

Now recall from §2 that the transformation

$$z = \left\{\frac{3}{2}\int_0^s P_0^{\frac{1}{2}}(s')\,ds'\right\}^{\frac{2}{3}} = \left\{\tfrac{3}{2}(w_c')^{-\frac{1}{2}}\int_0^s \surd\{V(s')\}\,ds'\right\}^{\frac{2}{3}} \tag{6.11}$$

maps the domain D_1 in the s-plane on to the domain D_2 in the z-plane, where D_2 is a fixed neighbourhood of a given fixed closed real interval $[z_a, z_b]$, with $z_a < 0 < z_b$. In §2 (*d*) the domain of interest in the z-plane was restricted to $D_3' = \operatorname{int}\bar{D}_3'$, where $\bar{D}_3' = \bar{\Omega} \cup \bar{S}'$, and $[z_a, z_b] \subset D_3' \subset D_2$. In the present (Orr–Sommerfeld) case we let the points z_a and z_b be the z-images of the points $s_a = -|y_c|$ and $s_b = |1-y_c|$. Suppose further that the inverse of the transformation (6.11) maps the sets Ω and S' in the z-plane on to the sets Ω_0' and S_0' (respectively) in the s-plane. Then, provided that the fixed number K_0 in the definition (6.6) of the disk Ω_0 is chosen sufficiently large, we have that
$$\Omega_0' \subset \Omega_0,$$

and so, if we define
$$\bar{D}_0' = \bar{\Omega}_0 \cup \bar{S}_0', \quad D_0' = \operatorname{int}\bar{D}_0',$$

it can be verified that
$$[s_a, s_b] \subset D_0' \subset D_1,$$

and also that the points $-y_c$, $1-y_c \in D_0'$ (for Im (c) sufficiently small).

As above, let $\tilde{v}$ and $\tilde{w}$ be Eagles's formally approximate solutions of $\mathscr{L}\Phi = 0$, and recall that $\chi_3^{(0)}$ is the leading term in the convergent series (6.2). Lemma 6.1 now enables us to prove

Lemma 6.2. (*a*) *There exists a function $v(.,\epsilon) \in \mathscr{H}(D_0')$ such that, for $|\epsilon|$ sufficiently small,*

(i) $|\tilde{v} - v| \leqslant M|\epsilon|^3$ *for* $s \in D_0'$, (6.12)

(ii) $|(d/ds)(\tilde{v} - v)| \leqslant M|\epsilon|^2$ *for* $s \in D_0'$, (6.13)

(iii) $$|\mathscr{L}v| \leqslant \begin{cases} M|\epsilon|^{-1} & \text{for} \quad s \in \Omega_0, \\ M|s|^{-1} & \text{for} \quad s \in D_0'\backslash\Omega_0, \end{cases} \tag{6.14}$$

(iv) $v|_{s=1-y_c} = V(1-y_c) + O(\epsilon^3)$, $(d/ds)\,v|_{s=1-y_c} = O(\epsilon^3)$, (6.15)

(v) $|v| \leqslant M$ *for* $s \in D_0'$. (6.16)

(*b*) *There exists a function $w(.,\epsilon) \in \mathscr{H}(D_0')$ such that, for $|\epsilon|$ sufficiently small and with G as in* (6.5),

(i) $|\chi_3^{(0)} - w| \leqslant M|\epsilon|$ *for* $s \in \Omega_0$,

$|\tilde{w} - w| \leqslant M|G(s,\epsilon)|\,|\epsilon|$ *for* $s \in D_0'\backslash\Omega_0$,

(ii) $|(\mathrm{d}/\mathrm{d}s)\,(\chi_3^{(0)}-w)| \leqslant M$ *for* $s \in \Omega_0$,
$|(\mathrm{d}/\mathrm{d}s)\,(\tilde{w}-w)| \leqslant M|G(s,\epsilon)|\,|\epsilon|^{\frac{3}{2}}|s|^{-\frac{5}{2}}$ *for* $s \in D_0' \backslash \Omega_0$,

$$\text{(iii)}\quad |\mathscr{L}w| \leqslant \begin{Bmatrix} M|\epsilon|^{-3} & \text{for} \quad s \in \Omega_0, \\ M|G(s,\epsilon)|\,|\epsilon|^{-\frac{9}{2}}|s|^{\frac{3}{2}} & \text{for} \quad s \in D_0',\, K_0|\epsilon| \leqslant |s| \leqslant |\epsilon|^{\frac{3}{4}}, \\ M|G(s,\epsilon)|\,|\epsilon|^{-\frac{3}{2}}\,|s|^{-\frac{5}{2}} & \text{for} \quad s \in D_0',\, |s| \geqslant |\epsilon|^{\frac{3}{4}}, \end{Bmatrix} \tag{6.17}$$

and (iv) $$(\mathrm{d}/\mathrm{d}s)\,w|_{s=1-y_c} = o(\epsilon^n), \quad (\mathrm{d}/\mathrm{d}s)^3\,w|_{s=1-y_c} = o(\epsilon^n) \quad \textit{for every} \quad n. \tag{6.18}$$

For the proof see appendix B.

Remark. It should be noted that Eagles used his composite series $\tilde{w}$ as a uniform approximation to Φ_3 *only* for $|s| \succ |\epsilon|$, that is, in the *intermediate* $(1 \succ |s| \succ |\epsilon|)$ and *outer* regions. In the inner region, $|s| \preccurlyeq |\epsilon|$, he was content to approximate Φ_3 by the function $\chi_3^{(0)}$. (Cf. his assumption (B) (i) as given in § 6 (*b*) above.)

(*d*) *The asymptotic solutions Φ_5 and Φ_3*

Suppose that the inverse of the transformation (6.11) maps the domain D_3 in the z-plane on to the domain D_0 in the s-plane. Then $[s_a, s_b] \subset D_0 \subset D_0'$, and the points $-y_c$, $1-y_c \in D_0$ (for Im (c) sufficiently small).

Our final theorem rigorously justifies the formal theory of Eagles.

Theorem 6.3. (*a*) *The equation $\mathscr{L}\Phi = 0$ has a solution Φ_5 in D_0 with the following properties for sufficiently small $|\epsilon|$.*

$$\text{(i)}\quad \Phi_5|_{s=1-y_c} = V(1-y_c), \quad (\mathrm{d}/\mathrm{d}s)\,\Phi_5|_{s=1-y_c} = 0, \tag{6.19}$$

$$\text{(ii)}\quad (\mathrm{d}/\mathrm{d}s)^3\,\Phi_5|_{s=1-y_c} = o(\epsilon^n) \quad \textit{for every} \quad n, \tag{6.20}$$

$$\text{(iii)}\quad |\Phi_5| \leqslant M \quad \textit{for} \quad s \in D_0. \tag{6.21}$$

Furthermore (with $\tilde{v}$ denoting Eagles's $E_{4,4}^{\mathrm{comp}}\,\Phi_5$),

$$\left.\begin{aligned} |\Phi_5-\tilde{v}| &\leqslant M|\epsilon|^3 \ln|\epsilon| \\ |(\mathrm{d}/\mathrm{d}s)\,(\Phi_5-\tilde{v})| &\leqslant M|\epsilon|^2 \end{aligned}\right\} \quad \textit{for} \quad s \in D_0.$$

(*b*) *The equation $\mathscr{L}\Phi = 0$ has a solution Φ_3 in D_0 such that*

$$(\mathrm{d}/\mathrm{d}s)\,\Phi_3|_{s=1-y_c} = o(\epsilon^n), \quad (\mathrm{d}/\mathrm{d}s)^3\,\Phi_3|_{s=1-y_c} = o(\epsilon^n)$$

for every n. Moreover (with $\chi_3^{(0)}$ as in (6.2), with $\tilde{w}$ denoting Eagles's $G\hat{E}_{0,0}^{\mathrm{comp}}(\Phi_3/G)$, and G as in (6.5)),

$$|\Phi_3-\chi_3^{(0)}| \leqslant M|\epsilon|^{\frac{3}{4}} \quad \textit{for} \quad s \in \Omega_0,$$

$$|\Phi_3-\tilde{w}| \leqslant M|G(s,\epsilon)|\,|\epsilon|^{\frac{3}{4}} \quad \textit{for} \quad s \in D_0 \backslash \Omega_0,$$

and $$|(\mathrm{d}/\mathrm{d}s)\,(\Phi_3-\chi_3^{(0)})| \leqslant M|\epsilon|^{-\frac{1}{4}} \quad \textit{for} \quad s \in \Omega_0,$$

$$|(\mathrm{d}/\mathrm{d}s)\,(\Phi_3-\tilde{w})| \leqslant M|G(s,\epsilon)|\,|\epsilon|^{\frac{3}{2}}|s|^{-\frac{5}{2}} \quad \textit{for} \quad s \in D_0 \backslash \Omega_0, \tag{6.22}$$

and for $|\epsilon|$ sufficiently small.

Proof. (*a*) The transformation (2.2 *a*, *b*):

$$z(s) = \left\{\tfrac{3}{2}(w_c')^{-\frac{1}{2}} \int_0^s \surd\{V(s')\}\,\mathrm{d}s'\right\}^{\frac{2}{3}}, \quad \psi(z) = (\mathrm{d}z/\mathrm{d}s)^{\frac{1}{2}}\,\Phi(s),$$

yields $$\mathscr{L}\Phi = (\mathrm{d}z/\mathrm{d}s)^{\frac{5}{2}}\,\mathscr{M}\psi;$$

we let z_* be the z-image of the point $s = 1-y_c$, and recall that $\lambda^{-2} = i\epsilon^3$. Theorem 4.1 now ensures that, corresponding to the function v described in lemma 6.2 (a), there exists a solution Φ_5 of $\mathscr{L}\Phi = 0$ satisfying the boundary conditions (6.19) and such that

$$|\Phi_5 - v| \leqslant M|\epsilon|^3 \ln|\epsilon|, \quad |(\mathrm{d}/\mathrm{d}s)(\Phi_5 - v)| \leqslant M|\epsilon|^2$$

for $s \in D_0$ and $|\epsilon|$ sufficiently small. The corresponding estimates for $\Phi_5 - \tilde{v}$ and its derivative then follow from lemma 6.2 (a).

Since $|v(s,\epsilon)| \leqslant M$ for $s \in D_0'$ by (6.16), we deduce from theorem 4.2 that Φ_5 has the uniform boundedness property (6.21), and that

$$\Phi_5 = \sum_{n=0}^{m} \epsilon^{3n}\Phi_{5,2n}(s) + R_m(s,\epsilon) \quad \text{in} \quad N^*,$$

where N_* is some small fixed neighbourhood of the point $s = 1-y_c$, and where all the functions $\Phi_{5,2n}$ belong to $\mathscr{H}(N_*)$. Also, for $j = 0, \ldots, 4$,

$$(\mathrm{d}/\mathrm{d}s)^j R_m(s,\epsilon) = O(\epsilon^{3m+3}\ln\epsilon) \quad \text{for} \quad |\epsilon| \to 0, s \in N_*,$$

and m is an arbitrary non-negative integer. Then, since

$$0 = \mathscr{L}\Phi_5 = \sum_{n=0}^{m} \epsilon^{3n}\mathscr{L}\Phi_{5,2n} + \mathscr{L}R_m,$$

we deduce that the functions $\Phi_{5,2n}$ $(n = 0, 1, \ldots, m)$ must satisfy the following differential equations in N_*:

$$V\{\Phi_{5,0}'' - k^2\Phi_{5,0}\} - V''\Phi_{5,0} = 0, \tag{6.23}$$

$$V\{\Phi_{5,2n}'' - k^2\Phi_{5,2n}\} - V''\Phi_{5,2n} = -(\mathrm{i}/w_c')\{\Phi_{5,2n-2}^{\mathrm{iv}} - 2k^2\Phi_{5,2n-2}'' + k^4\Phi_{5,2n-2}\} \quad (n = 1, 2, \ldots, m). \tag{6.24}$$

Also, by virtue of the boundary condition (6.19),

$$\Phi_{5,0}(1-y_c) = V(1-y_c), \quad \Phi_{5,0}'(1-y_c) = 0,$$
$$\Phi_{5,2n}(1-y_c) = 0, \quad \Phi_{5,2n}'(1-y_c) = 0 \quad (n = 1, 2, \ldots, m). \tag{6.25}$$

Differentiation of (6.23) yields

$$V\Phi_{5,0}''' = k^2V\Phi_{5,0}' - V'\{\Phi_{5,0}'' - k^2\Phi_{5,0}\} + V'''\Phi_{5,0} + V''\Phi_{5,0}'. \tag{6.26}$$

But, by condition (b) in § 6 (a) on the function $w(y) \equiv V(s) + c$,

$$V'(1-y_c) = V'''(1-y_c) = V^{\mathrm{v}}(1-y_c) = \ldots = 0.$$

Hence, using also (6.25) and the assumption $V(1-y_c) \neq 0$, we obtain from (6.26) that

$$\Phi_{5,0}'''(1-y_c) = 0.$$

Similarly, by differentiating (6.26) twice, we can show that $\Phi_{5,0}^{\mathrm{v}}(1-y_c) = 0$, and subsequently, by differentiation of (6.24), that

$$\Phi_{5,2n}'''(1-y_c) = 0 \quad (n = 1, 2, \ldots, m).$$

Thus $$(\mathrm{d}/\mathrm{d}s)^3\Phi_5|_{s=1-y_c} = (\mathrm{d}/\mathrm{d}s)^3 R_m|_{s=1-y_c} = O(\epsilon^{3m+3}\ln\epsilon),$$

and, since m is arbitrary, we have proved the result (6.20).

(b) We use the results of lemma 6.2 (b), and, arguing as in (a) above, appeal to theorem 5.1, which implies the existence of a solution Φ_3 of $\mathscr{L}\Phi = 0$, with properties as stated.

It should be pointed out that Eagles computed not only the composite series

$$G\hat{E}^{\text{comp}}_{0,0}(\Phi_3/G) \equiv \tilde{w},$$

but also the longer series $G\hat{E}^{\text{comp}}_{1,1}(\Phi_3|G)$, in order to check an additional term in the formal asymptotic expansion of Φ_3. However, he did this merely to compute the order of the error of the shorter series $\tilde{w}$, which is the one that he actually used to compute stability, and our analysis has in fact recovered rigorously this error estimate. (This follows from the fact that our error estimate (6.22) can be shown to imply precisely the estimate (6.19) in Eagles's (1969) paper.)

Appendix A. Solutions of a reference equation

Rabenstein (1958) considered the equation

$$u^{\text{iv}} + \lambda^2\{zu'' + \alpha u' + \beta u\} = 0, \tag{A 1}$$

where† α and β are complex parameters which depend on λ in such a way they are uniformly bounded for $|\lambda| \geq |\lambda_0|$, with $|\lambda_0|$ sufficiently large. It was assumed that $\beta \neq 0$.

A fundamental set of solutions $B_0(z;\alpha,\beta,\lambda)$, $A_1(z;\alpha,\beta,\lambda)$, $A_2(z;\alpha,\beta,\lambda)$, $B_3(z;\alpha,\beta,\lambda)$ of equation (A1) was then defined, and shown to have the following asymptotic properties for large $|\lambda|$ and z in the domain D_3' (as defined by (2.56)). The generic symbol $E(z,\lambda)$ is defined as in §2(*d*).

$$(a)\ \text{(i)}\ B_0 = \sum_{n=0}^{m} (1/n!)\,(\tfrac{1}{3}\lambda^{-2})^n\, b_{0,n}(z;\alpha+3n,\beta) + \lambda^{-2m-2}E(z,\lambda), \tag{A 2}$$

where the functions $b_{0,n}(z;\alpha+3n,\beta)$ $(n = 0, 1, \ldots, m)$ can be expressed in terms of a Bessel function of the first kind as

$$b_{0,n} = -2\pi\mathrm{i}\,\mathrm{e}^{-\pi\mathrm{i}(\alpha+3n)}\,(z^{\frac{1}{2}}\beta^{-\frac{1}{2}})^{1-\alpha-3n}J_{\alpha+3n-1}(2z^{\frac{1}{2}}\beta^{\frac{1}{2}});$$

$$\text{(ii)}\ A_1 = \begin{cases} \lambda^{\frac{2}{3}(\alpha-1)}E(z,\lambda) & \text{for}\quad z\in\Omega, \\ -\mathrm{i}\surd(\pi)\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}(\frac{1}{3}\alpha+\frac{1}{2})}\lambda^{\alpha-\frac{3}{2}}\,z^{\frac{1}{2}\alpha-\frac{5}{4}}\exp\{-\frac{2}{3}\mathrm{i}\lambda z^{\frac{3}{2}}\}\{1+\lambda^{-1}z^{-\frac{3}{2}}E(z,\lambda)\} & \text{for}\quad z\in D_3'\backslash\Omega; \end{cases} \tag{A 3}$$

$$\text{(iii)}\ A_2 = \begin{cases} \lambda^{\frac{2}{3}(\alpha-1)}E(z,\lambda) & \text{for}\quad z\in\Omega, \\ \surd(\pi)\,\mathrm{e}^{\pi\mathrm{i}(\frac{2}{3}\alpha-\frac{2}{3})}\lambda^{\alpha-\frac{3}{2}}\,z^{\frac{1}{2}\alpha-\frac{5}{4}}\exp\{\frac{2}{3}\mathrm{i}\lambda z^{\frac{3}{2}}\}\{1+\lambda^{-1}z^{-\frac{3}{2}}E(z,\lambda)\} & \text{for}\quad z\in D_3'\backslash\Omega; \end{cases} \tag{A 4}$$

$$\text{(iv)}\ B_3 = \left.\begin{cases} E(z,\lambda), & \operatorname{Re}(\alpha) \leq 1,\quad \alpha\neq 1 \\ \ln\lambda\, E(z,\lambda), & \alpha = 1 \\ \lambda^{\frac{2}{3}(\alpha-1)}E(z,\lambda), & \operatorname{Re}(\alpha) > 1 \end{cases}\right\}\quad \text{for}\quad z\in\Omega, \tag{A 5}$$

$$B_3 = \sum_{n=0}^{m}\frac{1}{n!}(\tfrac{1}{3}\lambda^{-2})^n\, b_{3,n}(z;\alpha+3n,\beta) + \lambda^{-2m-2}z^{-\alpha-2-3m}E(z,\lambda)\quad \text{for}\quad z\in D_3'\backslash\Omega, \tag{A 6}$$

where the functions $b_{3,n}\,(z;\alpha+3n,\beta)$ can be expressed in terms of a Hankel function as

$$b_{3,n} = \pi\mathrm{i}(\mathrm{e}^{2\pi\mathrm{i}}z^{\frac{1}{2}}\beta^{-\frac{1}{2}})^{1-\alpha-3n}H^{(1)}_{1-\alpha-3n}(2z^{\frac{1}{2}}\beta^{\frac{1}{2}}).$$

(*b*) Let $u(z;\alpha)$ denote any one of the functions B_0, A_1, A_2, B_3. Then

$$(\mathrm{d}/\mathrm{d}z)^j\,u(z;\alpha) = u(z;\alpha+j)\quad (j = 1, 2, \ldots). \tag{A 7}$$

(*c*) For the Wronskian of the functions B_0, A_1, A_2, B_3 we have in D_3' that

$$W(B_0, A_1, A_2, B_3) = 4\pi^2\,\mathrm{e}^{-2\pi\mathrm{i}\alpha}\beta^{\alpha-1}\lambda^{2\alpha+2}. \tag{A 8}$$

† Here the coefficients α and β of equation (A 1) are not necessarily the coefficients $\alpha, \beta \in A(2m)$ of $\mathscr{Q}u = 0$ as chosen in theorem 2.3.

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Appendix B. The modified Eagles functions v and w

Details of the formal approximation $\tilde{v}$

Eagles (1969) constructed the function

$$\tilde{v} \equiv E_{4,4}^{\text{comp}} \Phi_5 = \phi_5^{(0)}(y) + \epsilon\{v_2(\sigma) - a_1\sigma - b_1\sigma\ln\sigma\} + \epsilon^2\{v_4(\sigma) - a_2\sigma^2 - b_2\sigma^2\ln\sigma\}, \tag{B 1}$$

which can also be written in the form

$$\tilde{v} = \{\phi_5^{(0)}(y) + v_0(\sigma) - a_0 - a_1 s - a_2 s^2 - b_1 s\ln s - b_2 s^2 \ln s\} + \epsilon\ln\epsilon\, v_1(\sigma) + \epsilon\, v_2(\sigma) + \epsilon^2\ln\epsilon\, v_3(\sigma) + \epsilon^2 v_4(\sigma). \tag{B 2}$$

Here the function

$$\phi_5^{(0)}(y) = \phi_5^{(0)}(s+y_c) = \sum_{n=0}^{\infty} a_n s^n + \ln s \sum_{n=0}^{\infty} b_n s^n \tag{B 3}$$

is the leading term in the formal series ϕ_5 as given by (6.3), and satisfies the Rayleigh equation

$$V(s)\{\mathrm{d}^2\phi/\mathrm{d}s^2 - k^2\phi\} - V''(s)\,\phi = 0. \tag{B 4}$$

Writing $H \equiv w_c''/w_c'$ as before, we have in (B 3) that

$$b_1 = a_0 H, \quad b_2 = \tfrac{1}{2}a_0 H^2, \quad b_3 = \tfrac{1}{6}a_0 H(w_c'''/w_c' + k^2). \tag{B 5}$$

Furthermore, in (B 1) and (B 2), we have

$$\left.\begin{aligned} &v_0(\sigma) = a_0, \quad v_1(\sigma) = a_0 H\sigma, \quad v_2(\sigma) = a_0\chi_2^{(1)}(\sigma) - a_0 A_1 + (a_1 - a_0 B_1)\,\sigma, \\ &v_3(\sigma) = \tfrac{1}{2}a_0 H^2\sigma^2, \quad v_4(\sigma) = a_0(A_1^2 - A_2) - a_0 A_1\chi_2^{(1)}(\sigma) + a_0\chi_2^{(2)}(\sigma) \\ &\qquad\qquad + a_0(A_1 B_1 - B_2)\,\sigma + \tfrac{1}{2}(a_1 - a_0 B_1)\,H\sigma^2, \end{aligned}\right\} \tag{B 6}$$

where A_1, A_2, B_1 and B_2 are certain constants, and where the functions $\chi_2^{(1)}(\sigma)$, $\chi_2^{(2)}(\sigma)$ are those appearing in the convergent series (6.2) (which defines Φ_2), and satisfying

$$\left.\begin{aligned} &\frac{\mathrm{d}^4\chi_2^{(n)}}{\mathrm{d}\sigma^4} - \mathrm{i}\sigma\frac{\mathrm{d}^2\chi_2^{(n)}}{\mathrm{d}\sigma^2} = S^{(n)} \quad (n = 1, 2), \\ &\text{with} \quad S^{(1)} = -\mathrm{i}H, \quad S^{(2)} = \mathrm{i}\left\{\tfrac{1}{2}H\sigma^2\chi_2^{(1)''} - H\chi_2^{(1)} - \frac{w_c'''}{w_c'}\sigma - k^2\sigma\right\}. \end{aligned}\right\} \tag{B 7}$$

The form (B 1) of $\tilde{v}$ can also be written as

$$\tilde{v} = \phi_5^{(0)}(y) + \epsilon a_0 P_1(\sigma) + \epsilon^2 a_0\{Y(\sigma) - A_1 P_1(\sigma)\}, \tag{B 8}$$

where

$$\left.\begin{aligned} (\mathrm{d}/\mathrm{d}\sigma)^j P_1(\sigma) &= O(\sigma^{-2-j}) \\ (\mathrm{d}/\mathrm{d}\sigma)^j Y(\sigma) &= O(\sigma^{-1-j}) \end{aligned}\right\} \quad \text{for} \quad |\sigma| \to \infty, \ -\tfrac{7}{6}\pi + \delta \leqslant \arg\sigma \leqslant \tfrac{1}{6}\pi - \delta, \quad (j = 0, 1, 2). \tag{B 9}$$

Moreover, it can be shown that $P_1(\sigma)$ and $Y(\sigma)$ satisfy

$$\left.\begin{aligned} &\frac{\mathrm{d}^4 P_1}{\mathrm{d}\sigma^4} - \mathrm{i}\sigma\frac{\mathrm{d}^2 P_1}{\mathrm{d}\sigma^2} = -\frac{2H}{\sigma^3}, \\ &\frac{\mathrm{d}^4 Y}{\mathrm{d}\sigma^4} - \mathrm{i}\sigma\frac{\mathrm{d}^2 Y}{\mathrm{d}\sigma^2} = \frac{H^2}{\sigma^2} - \frac{2A_1 H}{\sigma^3} + \mathrm{i}H\left(\frac{\sigma^2}{2}\frac{\mathrm{d}^2 P_1}{\mathrm{d}\sigma^3} - P_1\right). \end{aligned}\right\} \tag{B 10}$$

Finally, note from (B 2) and (B 3) that another alternative form of $\tilde{v}$ is given by

$$\tilde{v} = \sum_{n=3}^{\infty} a_n s^n + \ln s \sum_{n=3}^{\infty} b_n s^n + v_0(\sigma) + \epsilon\ln\epsilon\, v_1(\sigma) + \epsilon v_2(\sigma) + \epsilon^2\ln\epsilon\, v_3(\sigma) + \epsilon^2 v_4(\sigma). \tag{B 11}$$

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Details of the formal approximation $\tilde{w}$

First, consider the transformation (6.11):

$$z = \left\{\tfrac{3}{2}(w_c')^{-\frac{1}{2}}\int_0^s \surd\{V(s')\}\,ds'\right\}^{\frac{2}{3}},$$

and introduce the stretching transformation $z = \epsilon\zeta$, whence

$$\zeta = \epsilon^{-1}(w_c')^{-\frac{1}{3}}\left\{\frac{3}{2}\int_0^s \surd\{V(s')\}\,ds'\right\}^{\frac{2}{3}}. \tag{B 12}$$

It then follows that

$$G(s,\epsilon) = \sigma^{-\frac{5}{4}}\exp\left(-\tfrac{2}{3}\,e^{\frac{1}{4}\pi i}\zeta^{\frac{3}{2}}\right) \tag{B 13}$$

by (6.5). Eagles (1969) constructed the function

$$\tilde{w} \equiv G\hat{E}^{\text{comp}}_{0,0}(\Phi_3/G) = G(s,\epsilon)\,\{\hat{f}_0(s) + \hat{\chi}_3^{(0)}(\sigma) - h_0\}, \tag{B 14}$$

where h_0 is a known non-zero complex constant, and where

$$\hat{f}_0(s) = h_0(w_c')^{\frac{5}{4}}s^{\frac{5}{4}}V^{-\frac{5}{4}}(s), \tag{B 15}$$

$$\hat{\chi}_3^{(0)}(\sigma) = \sigma^{\frac{5}{4}}\exp\left(\tfrac{2}{3}\,e^{\frac{1}{4}\pi i}\sigma^{\frac{3}{2}}\right)\chi_3^{(0)}(\sigma). \tag{B 16}$$

Here $\chi_3^{(0)}(\sigma)$ is the leading term in the convergent series (6.2) (which defines Φ_3), satisfies the equation

$$\frac{d^4\chi_3^{(0)}}{d\sigma^4} - i\sigma\frac{d^2\chi_3^{(0)}}{d\sigma^2} = 0, \tag{B 17}$$

and has the asymptotic expansion

$$\chi_3^{(0)}(\sigma) = h_0\exp\left(-\tfrac{2}{3}e^{\frac{1}{4}\pi i}\,\sigma^{\frac{3}{2}}\right)\left\{\sigma^{-\frac{5}{4}} - \frac{101}{48}\,e^{-\frac{3}{4}\pi i}\,\sigma^{-\frac{11}{4}} + \frac{35905}{2^9.3^2}\,e^{-\frac{1}{2}\pi i}\,\sigma^{-\frac{17}{4}} + O(\sigma^{-\frac{23}{4}})\right\}$$

$$\text{for}\quad |\sigma| \to \infty,\ -\tfrac{7}{6}\pi + \delta \leqslant \arg\sigma \leqslant \tfrac{1}{6}\pi - \delta. \tag{B 18}$$

Finally, note from (B 15) that

$$\hat{f}_0(s) - h_0 \sim -\tfrac{5}{8}h_0 Hs \quad \text{for} \quad |s| \to 0. \tag{B 19}$$

Proof of lemma 6.2

(*a*) Consider the formal approximation $\tilde{v}(s,\epsilon)$ in the form (B 11), let the function

$$u(\sigma) = \sigma + f(\sigma)$$

be as given in lemma 6.1, and construct the modified approximation

$$v(s,\epsilon) = \tilde{v}(s,\epsilon) + \ln\frac{\sigma + f(\sigma)}{\sigma}\sum_{n=3}^{\infty} b_n s^n \quad (s = \epsilon\sigma). \tag{B 20}$$

Then clearly $v(.,\epsilon) \in \mathscr{H}(D_0')$, and it remains to verify the properties (i) to (v).

(i) Consider the difference $v(s,\epsilon) - \tilde{v}(s,\epsilon)$ in the form

$$v(s,\epsilon) - \tilde{v}(s,\epsilon) = \sum_{n=3}^{\infty} b_n\epsilon^n\sigma^n\{\ln(\sigma + f(\sigma)) - \ln\sigma\}$$

if $s \in \Omega_0$ (which implies that $|\sigma| \leqslant K_0$), and in the form

$$v(s,\epsilon) - \tilde{v}(s,\epsilon) = \sum_{n=3}^{\infty} b_n\epsilon^n\sigma^n\ln(1 + \sigma^{-1}f(\sigma)) \tag{B 21}$$

if $s \in D_0' \backslash \Omega_0$. The result (6.12) then follows by virtue of the properties of the function

$$u(\sigma) = \sigma + f(\sigma).$$

(ii) The estimate (6.13) follows as in (i) above if we note that $(\mathrm{d}/\mathrm{d}s) = \epsilon^{-1}(\mathrm{d}/\mathrm{d}\sigma)$.

(iii) In this proof we shall write $(.)'$ for $(\mathrm{d}/\mathrm{d}s)(.)$, and $(\dot{.})$ for $(\mathrm{d}/\mathrm{d}\sigma)(.)$. Then from § 6(*a*), and the fact that $\lambda^2 = -\mathrm{i}\epsilon^{-3}$, we have the 'outer' and 'inner' forms of $\mathscr{L}$:

$$\mathscr{L}v \equiv v^{\mathrm{iv}} - 2k^2 v'' + k^4 v - \mathrm{i}(w_c')^{-1}\epsilon^{-3}[V(s)\{v'' - k^2 v\} - V''(s)\,v], \tag{B 22}$$

and

$$\mathscr{L}v \equiv \epsilon^{-4}(\ddot{v} - \mathrm{i}\sigma\dot{v}) + \mathrm{i}H\epsilon^{-3}(v - \tfrac{1}{2}\sigma^2\ddot{v}) + \epsilon^{-2}\left\{-2k^2\ddot{v} + \left(\frac{w_c'''}{w_c'} + k^2\right)\mathrm{i}\sigma v - \frac{1}{6}\frac{w_c'''}{w_c'}\sigma^3\ddot{v}\right\}$$
$$+\epsilon^{-1}\{O(\sigma^4)\,\ddot{v} + O(\sigma^2)\,v\} + k^4 v, \tag{B 23}$$

where the constant $H \equiv w_c''/w_c'$, and where the functions implied by the O-symbols are known explicitly.

First, suppose that $s \in \Omega_0$. Then, by (B 11) and (B 20),

$$v(s,\epsilon) = \sum_{n=3}^{\infty} a_n \epsilon^n \sigma^n + \sum_{n=3}^{\infty} b_n \epsilon^n \sigma^n \ln(\sigma + f(\sigma)) + \sum_{n=3}^{\infty} b_n \epsilon^n \sigma^n \ln \epsilon$$
$$+ v_0(\sigma) + \epsilon \ln \epsilon\, v_1(\sigma) + \epsilon v_2(\sigma) + \epsilon^2 \ln \epsilon\, v_3(\sigma) + \epsilon^2 v_4(\sigma),$$

where the $v_n(\sigma)$ are given by (B 6). Now compute $\mathscr{L}v$, with $\mathscr{L}$ in the inner form (B 23). Using (B 5) and (B 7), we find that (6.14) holds for $s \in \Omega_0$.

Next, suppose that $s \in D_0' \backslash \Omega_0$. We write

$$\mathscr{L}v = \mathscr{L}(v - \tilde{v}) + \mathscr{L}\tilde{v}$$

and apply $\mathscr{L}$, in the inner form (B 23), to the right-hand side of (B 21). This yields

$$|\mathscr{L}(v - \tilde{v})| \leqslant M|s|^{-1},$$

and it remains to bound $\mathscr{L}\tilde{v}$ similarly. To this end, we consider $\tilde{v}$ as given by (B 8). Then, using the properties (B 3) and (B 4) of $\phi_5^{(0)}$, and the outer form (B 22) of $\mathscr{L}$, we can show by means of a direct computation that

$$\mathscr{L}\phi_5^{(0)} = Ha_0(2s^{-3} - Hs^{-2}) + O(s^{-1}). \tag{B 24}$$

Also, if $\mathscr{L}$ is considered in the inner form (B 23), an explicit calculation yields

$$\mathscr{L}\{\epsilon a_0 P_1 + \epsilon^2 a_0(Y - A_1 P_1)\} = Ha_0\epsilon^{-3}\left(-\frac{2}{\sigma^3} + \epsilon\frac{H}{\sigma^2}\right) + O(s^{-1}), \tag{B 25}$$

where we have used the properties (B 9) and (B 10) of $P_1(\sigma)$ and $Y(\sigma)$. Now combine (B 24) and (B 25) to deduce from (B 8) that

$$|\mathscr{L}\tilde{v}| \leqslant M|s|^{-1} \quad \text{for} \quad s \in D_0' \backslash \Omega_0.$$

(iv) By (6.3) and (6.4) we have

$$\phi_5^{(0)}|_{s=1-y_c} = V(1-y_c), \quad (\mathrm{d}/\mathrm{d}s)\,\phi_5^{(0)}|_{s=1-y_c} = 0.$$

Now consider $\tilde{v}$ in the form (B 8). The result (6.15) then follows from (B 9), the bound (6.12) for $\tilde{v} - v$, and the estimate

$$|(\mathrm{d}/\mathrm{d}s)\{\tilde{v}(s,\epsilon) - v(s,\epsilon)\}||_{s=1-y_c} \leqslant M|\epsilon|^3$$

(which can be deduced from (B 21) if we use the property (6.9) of $f(\sigma)$, and recall that $|1-y_c|$ is bounded away from zero).

(v) Using (B 8), (B 9) and (B 11), we can show that $|\tilde{v}(s,\epsilon)| \leqslant M$ for $s \in D_0'$. The result (6.16) now follows if we bound the difference $\tilde{v}-v$ by (6.12).

(*b*) First, we note from (B 12) that

$$\zeta = \epsilon^{-1}(w_c')^{-\frac{1}{3}}\left\{\frac{3}{2}\int_0^s \surd\{V(s')\}\,\mathrm{d}s'\right\}^{\frac{2}{3}} = \sigma + O(\epsilon\sigma^2). \qquad \text{(B 26)}$$

Hence, if in lemma 6.1 we replace σ by ζ, it follows that the σ-image of the branch point $\zeta = 2\mathrm{i}K_0$ is arbitrarily close to the point $\sigma = 2\mathrm{i}K_0$. Now, by (B 13) to (B 16), we have

$$\tilde{w}(s,\epsilon) = \sigma^{-\frac{5}{4}}\exp\left(-\tfrac{2}{3}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\zeta^{\frac{3}{2}}\right)\{\hat{f}_0(s)-h_0\} + \exp\{\tfrac{2}{3}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}(\sigma^{\frac{3}{2}}-\zeta^{\frac{3}{2}})\}\,\chi_3^{(0)}(\sigma), \qquad \text{(B 27)}$$

from which we construct the modified approximation

$$w(s,\epsilon) = \{\sigma+f(\sigma)\}^{-\frac{5}{4}}\exp\{-\tfrac{2}{3}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}(\zeta+f(\zeta))^{\frac{3}{2}}\}\{\hat{f}_0(s)-h_0\}$$
$$+\exp[\tfrac{2}{3}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\{(\sigma+f(\sigma))^{\frac{3}{2}}-(\zeta+f(\zeta))^{\frac{3}{2}}\}]\,\chi_3^{(0)}(\sigma), \qquad \text{(B 28)}$$

with the function $u(\sigma) = \sigma+f(\sigma)$ as in lemma 6.1. Then clearly $w(.,\epsilon) \in \mathscr{H}(D_0')$, and it remains to verify the properties (i) to (iv).

(i), (ii) First, note from (B 19) that

$$\hat{f}_0(s)-h_0 = O(\epsilon\sigma). \qquad \text{(B 29)}$$

Then, using also (B 26), and the asymptotic property (B 16) of $\chi_3^{(0)}(\sigma)$ in the case $s \in D_0' \backslash \Omega_0$, we can show from (B 27) and (B 28) that the differences $\chi_3^{(0)}-w$ and $\tilde{w}-w$, as well as their first derivatives, are bounded as stated in the lemma.

(iii) Suppose that $s \in \Omega_0$. Then, using (B 26) and (B 29), we can show from (B 28) that

$$w(s,\epsilon) = \chi_3^{(0)}(\sigma) + O(\epsilon),$$

where the function implied by the O-symbol can be written explicitly. Now compute $\mathscr{L}w$, with $\mathscr{L}$ in the inner form (B 23), and use the fact that $\chi_3^{(0)}(\sigma)$ satisfies equation (B 17). We can then verify that (6.17) holds for $s \in \Omega_0$.

Next, suppose that $s \in D_0' \backslash \Omega_0$, write

$$\mathscr{L}w = \mathscr{L}\tilde{w} + \mathscr{L}(w-\tilde{w}), \qquad \text{(B 30)}$$

and consider $\mathscr{L}\tilde{w}$. By (B 27) we can write

$$\tilde{w}(s,\epsilon) = F_0(s,\epsilon) + F_1(\sigma,\epsilon),$$

where

$$F_0(s,\epsilon) = h_0(w_c')^{\frac{5}{4}}\epsilon^{\frac{5}{4}}\exp\left(-\tfrac{2}{3}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\zeta^{\frac{3}{2}}\right)V^{-\frac{5}{4}}(s),$$

$$F_1(\sigma,\epsilon) = \exp\{\tfrac{2}{3}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}(\sigma^{\frac{3}{2}}-\zeta^{\frac{3}{2}})\}\{\chi_3^{(0)}(\sigma) - h_0\sigma^{-\frac{5}{4}}\exp\left(-\tfrac{2}{3}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\sigma^{\frac{3}{2}}\right)\},$$

and where ζ is given in terms of $s(=\epsilon\sigma)$ by (B 26). A direct (but lengthy) computation now shows that

$$\mathscr{L}F_0 = h_0\exp\left(-\tfrac{2}{3}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\zeta^{\frac{3}{2}}\right)\left(\frac{101}{16}\mathrm{i}\,\epsilon^{-\frac{7}{4}}s^{-\frac{9}{4}} + \frac{101\times 11+45\times 13}{16\times 4}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\epsilon^{-\frac{1}{4}}s^{-\frac{15}{4}}\right.$$
$$\left.+\frac{45}{16}\times\frac{13}{4}\times\frac{17}{4}\times\epsilon^{\frac{5}{4}}s^{-\frac{21}{4}}\right) + R_0(s,\epsilon), \qquad \text{(B 31}a\text{)}$$

with

$$|R_0(s,\epsilon)| \leqslant M|G(s,\epsilon)|\,|\epsilon|^{-3}; \qquad \text{(B 31}b\text{)}$$

and

$$\mathscr{L}F_1 = h_0 \exp\left(-\tfrac{2}{3}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\zeta^{\frac{3}{2}}\right)\epsilon^{-4}\left(-\frac{101}{16}\mathrm{i}\sigma^{-\frac{9}{4}} - \frac{101\times 11 + 13\times 45}{16\times 4}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\sigma^{-\frac{15}{4}} - \frac{13}{4}\times\frac{45}{16}\times\frac{17}{4}\times\sigma^{-\frac{21}{4}}\right) + R_1(\sigma,\epsilon), \qquad \text{(B 32a)}$$

with

$$|R_1(\sigma,\epsilon)| \leqslant M|G(s,\epsilon)|\,|\epsilon|^{-3}|\sigma|^{\frac{3}{2}}; \qquad \text{(B 32b)}$$

here we have considered $\mathscr{L}$ in the outer form (B 22) to obtain (B 31), and in the inner form (B 23) to obtain (B 32). Then, since $\tilde{w} = F_0 + F_1$, we deduce that

$$|\mathscr{L}\tilde{w}| \leqslant M|G(s,\epsilon)|\,|\epsilon|^{-\frac{9}{2}}|s|^{\frac{3}{2}} \quad \text{for} \quad s\in D_0'\backslash\Omega_0. \qquad \text{(B 33)}$$

However, if $|s| \geqslant |\epsilon|^{\frac{3}{4}}$, we can improve on the estimate (B 33) if we observe that

$$F_1(\sigma,\epsilon) = h_0\exp\left(-\tfrac{2}{3}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\zeta^{\frac{3}{2}}\right)\left\{-\frac{101}{48}\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\epsilon^{\frac{11}{4}}s^{-\frac{11}{4}} + \frac{35905}{2^9\times 3^2}\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}}\epsilon^{\frac{17}{4}}s^{-\frac{17}{4}} + O\left(\epsilon^{\frac{23}{4}}s^{-\frac{23}{4}}\right)\right\}$$

by virtue of the asymptotic expansion (B 18) of $\chi_3^{(0)}(\sigma)$, and then apply $\mathscr{L}$ in the outer form (B 22) to F_1. This yields

$$\mathscr{L}F_1 = -\frac{101}{16}h_0\mathrm{i}\epsilon^{-\frac{7}{4}}s^{-\frac{9}{4}}\exp\left(-\tfrac{2}{3}\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\zeta^{\frac{3}{2}}\right) + R_2(s,\epsilon),$$

with

$$|\mathscr{R}_2(s,\epsilon)| \leqslant M|G(s,\epsilon)|\,|\epsilon|^{-\frac{3}{2}}|s|^{-\frac{5}{2}},$$

which, together with (B 31), implies that

$$|\mathscr{L}\tilde{w}| \leqslant M|G(s,\epsilon)|\,|\epsilon|^{-\frac{3}{2}}|s|^{-\frac{5}{2}} \quad \text{for} \quad s\in D_0'\backslash\Omega_0. \qquad \text{(B 34)}$$

Since it can be shown from (B 27) and (B 28) that $\mathscr{L}(w-\tilde{w})$ is also bounded by the right-hand sides of (B 33) and (B 34), the result (6.17) of the lemma now follows from (B 30) if we use the estimate (B 33) for $K_0|\epsilon| \leqslant |s| \leqslant |\epsilon|^{\frac{3}{4}}$, and the estimate (B 34) for $|s| \geqslant |\epsilon|^{\frac{3}{4}}$.

(iv) The result (6.18) is a direct consequence of the fact that the function $w(s,\epsilon)$ and its derivatives are exponentially small at $s = 1-y_c$.

Appendix C. Proof of lemma 6.1

By virtue of the remark (and its proof) after lemma 6.1, it suffices to prove that the function

$$q(\tau) = \tau - 1 + 2\mathrm{e}^{-\tau}$$

is bounded away from zero in S. We write $\tau = x+\mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}q(\tau) \equiv u+\mathrm{i}v$. Thus

$$u = \cos\theta(x-1+2\mathrm{e}^{-x}\cos y) + \sin\theta(y-2\mathrm{e}^{-x}\sin y)$$

and

$$\partial u/\partial r = \mathrm{Re}\{q'(\tau)\} = 1-2\mathrm{e}^{-x}\cos y.$$

Now consider any ray $y = cx$, $0 \leqslant c \leqslant 1$ in $\bar{S}$. Along this ray we have that

$$u = (c^2+1)^{-\frac{1}{2}}(x-1+2\mathrm{e}^{-x}\cos cx) + c(c^2+1)^{-\frac{1}{2}}(cx-2\mathrm{e}^{-x}\sin cx) \qquad \text{(C 1)}$$

and $\partial u/\partial r = 1-2\mathrm{e}^{-x}\cos cx$. Now suppose that $\partial u/\partial r = 0$ at $x = x_0$, $u = u_0$. Thus

$$\mathrm{e}^{x_0} = 2\cos cx_0, \qquad \text{(C 2)}$$

from which it is clear that $\partial u/\partial r$ has only one zero on $y = cx$. But $u|_{\tau=0} \geqslant 2^{-\frac{1}{2}}$ by (C 1), and so, if we can show that

$$u_0 \geqslant \delta_0 > 0 \qquad \text{(C 3)}$$

for some δ_0 independent of c, it will follow that u is bounded away from zero on the ray $y = cx$.

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It can be verified from (C2) that $0.5 < x_0 < 0.7$ if $0 \leqslant c \leqslant 1$. Also, by (C1) and (C2),

$$u_0 = (c^2+1)^{-\frac{1}{2}}\{(1+c^2)\,x_0 - 2c\,e^{-x_0}\sin cx_0\} = (c^2+1)^{-\frac{1}{2}}\{(1+c^2)\,x_0 - c\tan cx_0\}. \tag{C4}$$

Now consider the function

$$h(x) = (1+c^2)\,x - c\tan cx \tag{C5}$$

for $0 \leqslant x \leqslant 0.7$ and $0 \leqslant c \leqslant 1$. Clearly $h(0) = 0$, and

$$h'(x) = 1 - c^2\tan^2 cx \geqslant 1 - \tan^2(0.7) > 0.2.$$

Hence $h(x_0) > h\,(0.5) \geqslant \delta_1 > 0$, where δ_1 is independent of c, and (C3) follows from (C4) and (C5).

If $-1 \leqslant c \leqslant 0$, we set $c = -k$, and proceed as above to obtain a result similar to (C3). Thus u is bounded away from zero in S, and it follows that the same is true of $q(\tau)$.

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